Ising Model: Exercise sheet 2

Let $G = (V, E)$ be a finite graph and $\Omega = \{-1, 1\}^V$. Let $J = (J_e)_{e \in E}$ be a family of non-negative weights. We recall that for $\sigma \in \Omega$

$$H(\sigma) = -\sum_{xy \in E} J_{xy} \sigma_x \sigma_y.$$

Exercise 1. (*) Let $\beta > 0$. Set $V = \{0, 1\}^2$ and $E = \{xy : x, y \in V, \|x - y\|_1 = 1\}$. Set $J_e = \beta$ for all $e \in E$.

1. What are the possible values for H on Ω ?
2. For each of these values, how many different configurations take this value ?
3. For each $\sigma \in \Omega$, compute the measure μ_β.
4. Compute the probability $\mu_\beta(\sigma_{(0,0)} = \sigma_{(1,1)})$.
5. Compute the limit of μ_β as β goes to 0.
6. Compute the limit of μ_β as β goes to ∞.

Exercise 2. Let $d \geq 1$ and $n \geq 1$. Set $V = \{-n, \dots, n\}^d$ and $E = \{xy : x, y \in V, \|x - y\|_1 = 1\}$. Compute the minimum and the maximum of H on Ω.

Exercise 3. Let $A \subset V$ and $\sigma \in \Omega$. We define σ' as follows

$$\forall x \in V \qquad \sigma'_x = \begin{cases} -\sigma_x & \text{if } x \in A \\ \sigma_x & \text{otherwise.} \end{cases}$$

Compute $H(\sigma) - H(\sigma')$.

Exercise 4. Let $\beta \geq 0$ and $h \geq 0$. Set $\Lambda = \{-n, \dots, n\}^d$, $\Omega = \{-1, 1\}^\Lambda$ and $E = \{xy : x, y \in \Lambda, \|x - y\|_1 = 1\}$. For every $\sigma \in \Omega$, define

$$H^+(\sigma) = -\beta \sum_{xy \in E} \sigma_x \sigma_y - h \sum_{x : \|x\|_\infty = n} \sigma_x$$

and

$$Z^+ = \sum_{\sigma \in \Omega} \exp(-H^+(\sigma)).$$

We define the measure $\widetilde{\mu}$ on Ω as follows

$$\forall \sigma \in \Omega \quad \widetilde{\mu}(\sigma) = \frac{1}{Z^+} \exp(-H^+(\sigma)).$$

Let $g \notin \Lambda$ be a ghost point. Set $E' = E \cup \{xg : x \in \Lambda, \|x\|_\infty = n\}$. Let μ be the Ising measure associated with the finite graph $(V \cup \{g\}, E')$ and the weights $J_e = \beta$ for every $e \in E$ and $J_e = h$ for $e \in E' \setminus E$. Set $\mu^+ = \mu[\,\cdot \,|\, \sigma_g = +1]$.

Let $\sigma \in \Omega$. Let $\overline{\sigma} \in \{-1, 1\}^{\Lambda \cup \{g\}}$ be defined as follows

$$\forall x \in \Lambda \cup \{g\} \qquad \overline{\sigma}_x = \begin{cases} \sigma_x & \text{if } x \in \Lambda \\ 1 & \text{if } x = g. \end{cases}$$

Prove that $\widetilde{\mu}(\sigma) = \mu^+(\overline{\sigma})$.