Ising Model: Exercise sheet 3

Let $G = (V, E)$ be a finite graph and $\Omega = \{-1, 1\}^V$. Let $J = (J_e)_{e\in E}$ be a family of non-negative coupling constants. We recall that for $\sigma \in \Omega$

$$H(\sigma) = -\sum_{xy\in E} J_{xy}\sigma_x\sigma_y.$$

Exercise 1. ($\star$) Let $G = (\mathbb{Z}/2n\mathbb{Z}, E)$ be the cyclic graph that is $E = \{\{i, i+1\} : 1 \le i \le 2n\}$. We assume that for all $e \in E$, $J_e = \beta$. Compute the following quantities using the random-current representation:

1. $\langle\sigma_1\sigma_3\rangle$
2. $\langle\sigma_{\{1,\dots,n\}}\rangle$
3. $\langle\sigma_{\{2k\,:\,1\le k\le n\}}\rangle$.

Exercise 2 (High temperature expansion). ($\star$) Set $J_e = \beta$ for all $e \in E$ with $\beta > 0$.

1. (a) By using the identity

$$\forall x \in \{-1, 1\} \quad e^{\beta x} = \cosh(\beta)(1 + x\tanh(\beta)),$$

prove that for $\sigma \in \Omega$

$$e^{-\beta H(\sigma)} = \cosh(\beta)^{|E|} \sum_{\eta\in\{0,1\}^E} \tanh(\beta)^{|\eta|} \prod_{xy:\,\eta_{xy}=1} \sigma_x\sigma_y,$$

where for $\eta \in \{0, 1\}^E$, $|\eta| = \sum_{e\in E}\eta_e$.

(b) Prove that for $A \subset \Lambda$

$$\langle\sigma_A\rangle = \frac{\sum_{\eta:\partial\eta=A}\tanh(\beta)^{|\eta|}}{\sum_{\eta:\partial\eta=\emptyset}\tanh(\beta)^{|\eta|}}. \tag{1}$$

2. Let N be a *ppp*(J) on E. Set $\eta = (\mathbf{1}_{N_e \text{ is odd}})_{e\in E}$.

(a) Prove that $\partial N = \partial\eta$.

(b) Compute the probability $\mathbb{P}(N_e \text{ is odd})$ for $e \in E$.

(c) Using the random-current representation, prove formula (1).

Exercise 3. Let $G = (V, E)$ be a graph. We assume that for all $e \in E$, $J_e > 0$. The aim of this exercise is to find the sets A that satisfy $\langle\sigma_A\rangle > 0$ using the random-current representation.

1. Write $V = \cup_{1\le i\le m} V_i$ where $(V_i,\ 1 \le i \le m)$ are the connected components of G. Prove that for any current $\mathbf{n} \in \mathbb{N}^E$, for any $i \in \{1, \dots, m\}$ the number of source in V_i $|\partial\mathbf{n} \cap V_i|$ is even.
2. Let $A \subset V$. Prove that if A intersects a connected component of G an odd number of times then $\langle\sigma_A\rangle = 0$.
3. Let $A \subset V$ such that each connected component of G has an even intersection with A. By pairing points in A, build a deterministic current $\mathbf{n}$ such that $\partial\mathbf{n} = A$. Deduce that $\langle\sigma_A\rangle > 0$.

Exercise 4. Let $\mathbf{n}$ be a sourceless current on (V, J). Let $x, y \in V$ such that x and y are connected in $\mathbf{n}$. Prove that x and y are doubly connected, that is there exist two paths γ_1 and γ_2 joining x and y such that

$$\forall e \in E \quad \mathbf{1}_{e\in\gamma_1} + \mathbf{1}_{e\in\gamma_2} \le \mathbf{n}_e\,.$$