Ising Model: Exercise sheet 4

Let $G = (V, E)$ be a finite graph and $\Omega = \{-1, 1\}^V$. Let $J = (J_e)_{e \in E}$ be a family of non-negative coupling constants. For $A \subset V$ we consider the set of currents

$$\mathcal{F}_A = \{\mathbf{n} \in \mathbb{N}^E \,:\, |C \cap A| \text{ is even for every cluster } C \text{ of } \mathbf{n}\}$$

Exercise 1. ($\star$)

1. Prove that if $\partial \mathbf{n} = A$ then $\mathbf{n} \in \mathcal{F}_A$.
2. Prove that if $\mathbf{m} \leq \mathbf{n}$ and $\mathbf{m} \in \mathcal{F}_A$ then $\mathbf{n} \in \mathcal{F}_A$.
3. Prove that for $\mathbf{n} \in \mathcal{F}_A$, there exists $\eta \in \{0, 1\}^E$ such that $\eta \leq \mathbf{n}$ and $\partial \eta = A$.
4. Let $A, B, C \subset V$ and M, N two independent *ppp*(J), prove the following switching lemma

$$\mathbb{P}[\partial M = A,\, \partial N = B,\, M + N \in \mathcal{F}_C] = \mathbb{P}[\partial M = A \Delta C,\, \partial N = B \Delta C,\, M + N \in \mathcal{F}_C]$$

Exercise 2. For $S \subset V$ and $\mathbf{n} \in \mathbb{N}^E$ we define $\mathbf{n}^S$ as follows

$$\forall e \qquad \mathbf{n}^S_e = \begin{cases} \mathbf{n}_e & \text{if } e \subset S \\ 0 & \text{otherwise.} \end{cases}$$

In this exercise, we aim to prove that if M, N are independent *ppp*(J), x, y, z are distinct points in V, we have

$$\mathbb{P}[\partial M^S = \emptyset,\, \partial N = xz,\, x \overset{M^S + N^S}{\longleftrightarrow} y] = \mathbb{P}[\partial M^S = xy,\, \partial N = yz]. \qquad (1)$$

1. Prove that we have

$$\mathbb{P}[\partial M^S = \emptyset,\, \partial N = xz,\, x \overset{M^S + N^S}{\longleftrightarrow} y] = \sum_{\mathbf{k} \in \mathbb{N}^E} \mathbb{P}[\partial M^S = \emptyset,\, \partial N^S = \partial \mathbf{k} \Delta xz,\, x \overset{M^S + N^S}{\longleftrightarrow} y]\, \mathbb{P}[N - N^S = \mathbf{k}].$$

2. Using the switching lemma in the previous formula, prove that we have (1).