Ising Model: Exercise sheet 5

Let $G = (V, E)$ be a finite graph and $\Omega = \{-1, 1\}^V$. Let $J = (J_e)_{e \in E}$ be a family of non-negative coupling constants.

Exercise 1. Let $(\alpha_S)_{S \subset V}$ such that for all $S \subset V$ $\alpha_S \geq 0$. Let $A \subset V$. Prove that

$$\Big\langle \sinh(\sigma_A) \cdot \cosh(\sum_{S \subset V} \alpha_S \sigma_S) \Big\rangle \geq \Big\langle \sinh(\sigma_A) \Big\rangle \Big\langle \cosh(\sum_{S \subset V} \alpha_S \sigma_S) \Big\rangle .$$

Exercise 2. ($\star$) [Monotonicity in the weights, alternative proof] In this exercise, we aim to prove the monotonicity of the expectation with respect to the weights J. To avoid confusion, we will denote by $\langle \cdot \rangle_J$ the expectation for the Ising measure with weights J.

1. Let $e = xy \in E$. Prove that for any $A \subset V$ we have

$$\frac{\partial}{\partial J_e} \langle \sigma_A \rangle_J = \langle \sigma_A \sigma_{xy} \rangle_J - \langle \sigma_A \rangle_J \langle \sigma_{xy} \rangle_J .$$

2. Conclude that if $J \leq J'$ then for any $A \subset V$, $\langle \sigma_A \rangle_{J'} \geq \langle \sigma_A \rangle_J$.

Exercise 3. ($\star$) Let $A \subset V$. Prove that $\mu_J[\forall x, y \in A\ \sigma_x = \sigma_y]$ is non-decreasing in J.

Exercise 4. In this exercise, we aim to find sufficient and necessary conditions on $x, y, z, t \in V$ with $x \neq y$ and $z \neq t$ to have

$$\langle \sigma_{xy} \sigma_{zt} \rangle - \langle \sigma_{xy} \rangle \langle \sigma_{zt} \rangle > 0. \tag{1}$$

Prove that (1) holds if and only if there exist two disjoint paths from $\{x, y\}$ and $\{z, t\}$ in G.