Ising Model: Exercise sheet 6

Let $G = (V, E)$ be a finite graph and $\Omega = \{-1, 1\}^V$. Let $J = (J_e)_{e \in E}$ be a family of non-negative coupling constants.

Exercise 1. To avoid confusion, we will denote by $\langle \cdot \rangle_G$ the expectation for the Ising measure on the graph G. Let $A \subset V$. Let K be the set of vertices of G that are connected to A in $G[J > 0]$. The aim of this exercise is to prove that

$$\langle \sigma_A \rangle_G = \langle \sigma_A \rangle_K$$

1. Prove that
$$H_G(\sigma) = H_K(\sigma|_K) + H_{V \setminus K}(\sigma|_{V \setminus K}).$$

2. Conclude.

Exercise 2. ($\star$) (Pushing boundary conditions) Let $A \subset V$. Let $f \in E$ such that $f \cap A = \emptyset$. Let us denote by $\overline{G}$ and $\overline{J}$ the graph and the coupling constants obtained from (G, J) by identifying endpoints of f as one vertex. More precisely, $\overline{G}$ has vertex set $\overline{V} = (V \setminus f) \cup \{v\}$ (where $v \notin V$ is an additional vertex), and edge set $\overline{E} = \{e \in E \;:\; e \cap f = \emptyset\} \cup \{xv, \;:\; x \notin f, x \text{ neighbour to one vertex of } f\}$.

We define the following families of coupling constants $J^{(\alpha)}$ for $\alpha \geq J_f$ as follows:

$$\forall e \in E \qquad J_e^{(\alpha)} = \begin{cases} \alpha & \text{if } e = f \\ J_e & \text{otherwise} \end{cases}.$$

1. Prove that $\langle \sigma_A \rangle_{\overline{G}, \overline{J}} = \lim_{\alpha \to \infty} \langle \sigma_A \rangle_{G, J^{(\alpha)}}$.

2. We recall that $\Lambda_n = \{-n, \ldots, n\}^d$. Use question 1 to prove that for every $A \subset \Lambda_n$, we have $\langle \sigma_A \rangle^+_{\Lambda_n} \geq \langle \sigma_A \rangle^+_{\Lambda_{n+1}}$.

Let $\Lambda \subset\subset \mathbb{Z}^d$. Let $\Omega_\Lambda = \{-1, 1\}^V$. For $\sigma \in \Omega_\Lambda$ we recall that

$$H^+_\Lambda(\sigma) = -\beta \sum_{\substack{xy \subset \Lambda: \\ x \sim y}} \sigma_x \sigma_y - \beta \sum_{\substack{x \in \Lambda, y \notin \Lambda: \\ x \sim y}} \sigma_x.$$

Exercise 3. Let $A \subset S \subset \Lambda$. We aim to prove that

$$\langle \sigma_A \rangle^+_S = \langle \sigma_A | \forall x \in \Lambda \setminus S \; \sigma_x = 1 \rangle^+_\Lambda.$$

Let $F = \{e \in E \,:\, e \cap S = \emptyset,\, e \cap \Lambda \setminus S \neq \emptyset\}$. Let $\sigma \in \Omega_\Lambda$ such that $\sigma_x = 1$ for all $x \in \Lambda \setminus S$.

1. Prove that
$$H^+_\Lambda(\sigma) = H^+_S(\sigma|_S) - \beta |F|.$$

2. Conclude.