Ising Model: Exercise sheet 7

We set $\Lambda_n = \{-n, \dots, n\}^2$ and E_n be the following set of edges

$$E_n = \{xy : x, y \in \Lambda_n, x \sim y\} \cup \{xy : x \in \Lambda_n, y \notin \Lambda_n, x \sim y\}.$$

We define the dual graph of $\mathbb{Z}^2$ by associating to each edge $e \in E(\mathbb{Z}^2)$ the unique edge e^* in $E((1/2, 1/2) + \mathbb{Z}^2)$ that crosses e. We define

$$\Omega^+_{\Lambda_n} = \{\sigma \in \{-1, 1\}^{\Lambda_n \cup \partial\Lambda_n} : \forall x \in \partial\Lambda_n \quad \sigma_x = 1\}.$$

We define (Λ_n^*, E_n^*) the dual graph of (Λ_n, E_n). We define

$$\mathcal{E}^* = \{\mathbf{n} \in \{0, 1\}^{E_n^*} : \mathbf{n} \text{ is a sourceless current}\}.$$

Exercise 1 (Low temperature expansion). We define the following map ϕ that associates to each $\sigma \in \Omega^+_{\Lambda_n}$ the following current $(\mathbf{1}_{\sigma_e=-1})_{e^* \in E_n^*}$. That is to say, to each edge $e^* \in E_n^*$ we associate the value 1 if the spins of σ disagree on the dual edge e.

1. Prove that for any $\sigma \in \Omega^+_{\Lambda_n}$, the current $\phi(\sigma)$ is sourceless.
2. Prove that for $\sigma, \tau \in \Omega^+_{\Lambda_n}$ such that $\sigma \neq \tau$, we have $\phi(\tau) \neq \phi(\sigma)$. Conclude that ϕ is injective.
3. In this question we aim to prove that ϕ is surjective. Let $\eta \in \mathcal{E}^*$. Let $x = (x_1, x_2) \in \Lambda_n$. We define N_x^1 (respectively N_x^2) be the number of edges in e^* such that $\eta_{e^*} = 1$, crossed by the line $[-n-1, x_1] \times \{x_2\}$ (respectively $\{x_1\} \times [-n-1, x_2]$).
   (a) By summing the degrees of vertices in $\Lambda_n^* \cap [-n-1, x_1] \times [-n-1, x_2]$ prove that N_x^1 and N_x^2 have the same parity.
   (b) Let $\sigma = ((-1)^{N_x^1})_{x \in \Lambda_n \cup \partial\Lambda_n}$. Prove that $\phi(\sigma) = \eta$.
   (c) Let $\eta = \phi(\sigma)$. Prove that $\mu^+_{\Lambda_n,\beta}(\eta) = \exp(-2\beta|\eta|)/Z$ where $Z = \sum_{\eta \in \mathcal{E}^*} \exp(-2\beta|\eta|)$.

Exercise 2 (Peierls argument). We say that there exists a loop in E_n^* for $\phi(\sigma)$ surrounding 0 if any path from 0 to $\partial\Lambda_n$ has an edge e such that $\phi(\sigma)_{e^*} = 1$.

1. Prove that for $\sigma \in \Omega^+_{\Lambda_n}$, if $\sigma_0 = -1$ then there exists at least one loop surrounding 0.
2. Let γ be a loop of E_n^* surrounding 0. Prove using the low temperature expansion that
$$\mu^+_{\Lambda_n,\beta}[\gamma \subset \phi(\sigma)] \leq \exp(-2\beta|\gamma|)$$
3. Prove that for β large enough $\mu^+_{\Lambda_n,\beta}[\sigma_0 = -1] \leq \frac{1}{4}$.
4. Conclude that $\beta_c(2) < \infty$.

Exercise 3 (High temperature expansion). ($\star$) The aim of this exercise is to prove that $\beta_c(d) > 0$. Let $\Lambda_n = \{-n, \dots, n\}^d$ and g be the ghost. Let $G = (\Lambda_n \cup \{g\}, E)$ and $J = (J_e)_{e \in E}$ where

$$E = \{xy \subset \Lambda_n : x \sim y\} \cup \{xg : \exists y \in \mathbb{Z}^d \setminus \Lambda_n \quad x \sim y\},$$

$J_e = \beta$ for $e \subset \Lambda_n$ and for $x \in \partial\Lambda_n$, $J_{xg} = \sum_{y \notin \Lambda_n : x \sim y} \beta$.

1. Prove that
$$\langle \sigma_0 \rangle^+_{\Lambda_n} = \frac{\mathbb{P}[\partial N = \{0,g\}]}{\mathbb{P}[\partial N = \emptyset]}$$
where N is a $ppp(J)$.

2. Let $\mathbf{n} \in \mathbb{N}^E$ such that $\partial \mathbf{n} = \{0, g\}$. Prove that there exists a self-avoiding path γ between 0 and g in $G[\mathbf{n} > 0]$. Deduce that we can write $\mathbf{n} = \mathbf{m}^{E\setminus\gamma} + \mathbf{m}^\gamma$ where $\partial \mathbf{m}^{E\setminus\gamma} = \emptyset$ and the current $\mathbf{m}^{E\setminus\gamma}$ is null on γ, the current $\mathbf{m}^\gamma$ has its support in γ and $\partial \mathbf{m}^\gamma = \{0, g\}$.

3. Set Γ_m be the set of self-avoiding path from 0 to g of length m. Prove that
$$\mathbb{P}[\partial N = \{0,g\}] \leq \sum_{m \geq n} \sum_{\gamma \in \Gamma_m} \mathbb{P}[\forall e \in \gamma \ N_e \text{ odd}] \mathbb{P}[\partial N^{E\setminus\gamma} = \emptyset].$$

4. Let $\gamma \in \Gamma_m$. Prove that
$$\mathbb{P}[\partial N^{E\setminus\gamma} = \emptyset] \prod_{e \in \gamma} \cosh(J_e) e^{-J_e} \leq \mathbb{P}[\partial N = \emptyset]$$

5. Deduce that
$$\langle \sigma_0 \rangle^+_{\Lambda_n} \leq \sum_{m \geq n} \sum_{\gamma \in \Gamma_m} \tanh^m(d\beta)$$

6. Prove that for β small enough we have
$$\lim_{n \to \infty} \langle \sigma_0 \rangle^+_{\Lambda_n} = 0.$$
Conclude.