Ising Model: Exercise sheet 9

Exercise 1. ($\star$) For a real-valued random variable X, we define its inverse distribution function F_X^{-1} as follows

$$\forall p \in [0,1] \qquad F_X^{-1}(p) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\}.$$

1. Let U be a uniform random variable on $[0,1]$. Prove that $F_X^{-1}(U)$ has the same law as X.
2. Let X, Y be two real-valued random variables. Prove that if for all $p \in [0,1]$, $F_X^{-1}(p) \leq F_Y^{-1}(p)$ then X is stochastically dominated by Y.
3. Prove that X is stochastically dominated by Y in the following cases.
   (a) X and Y are exponential random variables of parameter λ and μ with $\lambda < \mu$.
   (b) $Y = |X|$ and X is any real-valued random variable.
   (c) X has distribution $\frac{\alpha}{x^{\alpha+1}}\mathbf{1}_{x\geq 1}dx$, Y has distribution $\frac{\beta}{x^{\beta+1}}\mathbf{1}_{x\geq 1}dx$ with $0 < \alpha \leq \beta$.

Exercise 2. Let X be a random variable on $\{0,1\}^2$. Give necessary and sufficient conditions on the distribution of X for X to stochastically dominate the uniform measure on $\{0,1\}^2$ where we use the lexicographic order on $\mathbb{R}^2$.

Exercise 3. ($\star$) Let V be a finite a set. We consider the Glauber dynamics for a positive measure μ on $\Omega = \{-1,1\}^V$. Let $X := (X_n)_{n\geq 0}$ be the corresponding Markov chain.

1. Prove that X is aperiodic, irreducible.
2. Prove that μ is the unique invariant measure for this Markov Chain.

   Hint: Shows that it is reversible.

Exercise 4. Let V be a finite set. Let $p \in (0,1)$. Prove that the measure $\mathbb{P}_p = (\mathsf{Bernoulli}(p))^{\otimes V}$ satisfies FKG.

Exercise 5. Let $n \geq 1$, let $\Lambda_n = \{-n, \ldots, n\}^d$. Let $\beta > 0$. Let μ be the Ising measure on Λ_n at inverse temperature β without boundary conditions and μ^+ the Ising measure with plus boundary conditions. Prove that μ^+ stochastically dominates μ.