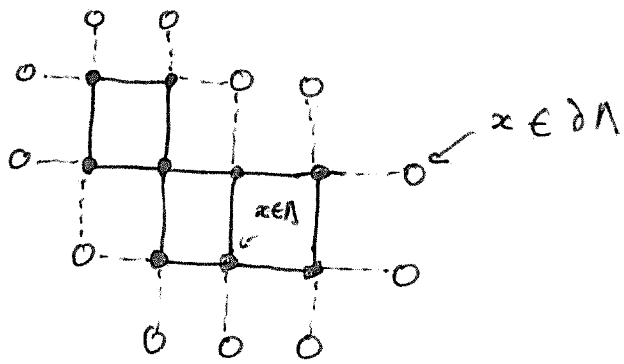
CHAPTER 7

FINITE-VOLUME ISING MEASURES

WITH CONFIGURATIONAL BONDARY CONDITIONS

Framework.

- $\Lambda \subset\subset \mathbb{Z}^d$, $\partial\Lambda = \{ x \in \mathbb{Z}^d \setminus \Lambda : \exists y \in \Lambda \; x \sim y \}$



- $\beta \geq 0$
- spin configuration $\sigma \in \Omega = \{+1, -1\}^\Lambda$

Goals:

- def. config. Ising measures on Ω
- prepare for infinite volume measures
- compare with mono-chromatic / free b.c.

1 DEFINITIONS.

Def. boundary conditions (bc.) for Λ : $\omega \in \{+1,-1\}^{\partial\Lambda}$

Def. The Ising. measure in Λ with bc. ω and inverse temperature β is def. by

$$\forall \sigma \in \Omega \qquad \mu^{\omega}[\sigma] = \frac{1}{Z^{\omega}} e^{-H^{\omega}(\sigma)}$$

where . $H^{\omega}(\sigma) = -\beta \sum_{\substack{xy \subset \Lambda \\ x\sim y}} \sigma_x \sigma_y - \beta \sum_{\substack{x\in\Lambda, y\in\partial\Lambda \\ x\sim y}} \sigma_x \omega_y$

. $Z^{\omega} = \sum_{\sigma\in\Omega} e^{-H^{\omega}(\sigma)}$.

Not. $\mu^{\omega} = \mu^{\omega}_{\Lambda,\beta} = \mu^{\omega}_{\Lambda} = \mu^{\omega}_{\beta}$

$$Z^{\omega}[f] = \sum_{\sigma\in\Omega} f(\sigma)\, e^{-H^{\omega}(\sigma)}$$

$$\langle f \rangle^{\omega} = \sum_{\sigma\Omega} f(\sigma)\, \mu^{\omega}[\sigma]$$

2 COMPARISON BETWEEN B.C.

Prop. If $\omega \leq \omega'$, then $\mu^{\omega} \ll \mu^{\omega'}$

In particular $\forall \omega$ b.c. $\mu^{-} \ll \mu^{\omega} \ll \mu^{+}$

<u>Application</u> : $\langle \sigma_0 \rangle^{\omega} << \langle \sigma_0 \rangle^{\omega'}$ if $\omega \leq \omega'$.

<u>Proof</u>: Let $\sigma \leq \sigma'$. ($\sigma^{(z)}$, $\sigma_{(z)}$ config. with $+1/-1$ at z)
Let $z \in \Lambda$

$$\frac{\mu^{\omega}[\sigma^{(x)}]}{\mu^{\omega}[\sigma_{(x)}]} = e^{-H^{\omega}(\sigma^{(z)}) + H^{\omega}(\sigma_{(z)})}$$

$$= \exp\left(2\beta \sum_{\substack{y \in \Lambda \\ y \sim z}} \sigma_y + 2\beta \sum_{\substack{y \in \partial\Lambda \\ y \sim z}} \omega_y \right)$$

$$\leq \exp\left(2\beta \sum_{\substack{y \in \Lambda \\ y \sim z}} \sigma'_y + 2\beta \sum_{\substack{y \in \partial\Lambda \\ y \sim z}} \omega'_y \right)$$

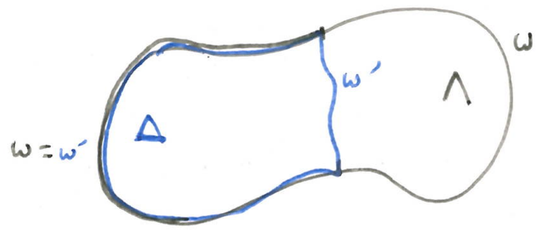
$$= \frac{\mu^{\omega'}[\sigma'^{(z)}]}{\mu^{\omega'}[\sigma'_{(z)}]}$$

By Holley criterion $\mu^{\omega} << \mu^{\omega'}$ $\boxtimes$

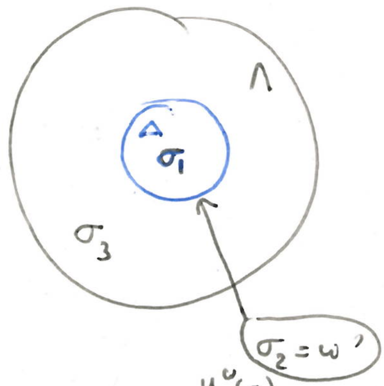
3 DOMAIN MARKOV PROPERTY.

<u>Prop.</u> Let $\Delta \subset \Lambda$, ω' bc for Δ compatible with ω (i.e. $\forall x \in \partial\Delta \cap \partial\Lambda$ $\omega'(x) = \omega(x)$). We have

$$\forall \eta \in \Omega_\Delta \quad \mu_\Lambda^\omega[\sigma_{|\Delta} = \eta \mid \forall x \in \partial\Delta \cap \Lambda \ \sigma_x = \omega'_x] = \mu_\Delta^{\omega'}[\eta]$$



<u>Proof</u>: For simplicity assume $\partial\Delta \subset \Lambda$



We decompose each config. $\sigma \in \Omega_\Lambda$ into $\sigma_1 \in \Omega_\Delta$, $\sigma_2 \in \Omega_{\partial\Delta}$, $\sigma_3 \in \Omega_{\Lambda \setminus \bar\Delta}$ ($\bar\Delta = \Delta \cup \partial\Delta$)

$$\text{LHS} = \frac{\mu_\Lambda[\sigma_1 = \eta, \sigma_2 = \omega']}{\mu_\Lambda[\sigma_2 = \omega']} = \frac{\sum_{\sigma:\sigma_1=\eta,\ \sigma_2=\omega'} e^{-H^\omega(\sigma)}}{\sum_{\sigma:\sigma_2=\omega'} e^{-H^\omega(\sigma)}}$$

$$\forall \sigma \text{ s.t. } \sigma_2 = \omega', \quad H^\omega(\sigma) = H_\Delta^{\omega'}(\sigma_1) + H^{\omega,\omega'}_{\Lambda\setminus\bar\Delta}(\sigma_3) \underbrace{- \beta \sum_{\substack{xy \subset \partial\Delta \\ x \sim y}} \omega'_x \omega'_y}_{C(\omega')}$$

Hence

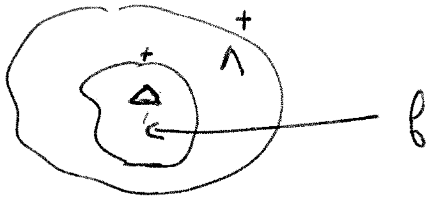
$$\text{LHS} = \frac{e^{-C(\omega')} \times e^{-H_\Delta^{\omega'}(\eta)} \times Z^{\omega,\omega'}_{\Lambda\setminus\bar\Delta}}{e^{-C(\omega')} \cdot Z_\Delta^{\omega'} \times Z^{\omega,\omega'}_{\Lambda\setminus\bar\Delta}} = \mu_\Delta^{\omega'}[\eta] \qquad \blacksquare$$

Exercise:

Let $\Delta \subset \Lambda$. Let f be an increasing function of $(\sigma_x)_{x\in\Delta}$.

We have

$$\langle f \rangle^+_\Lambda \leq \langle f \rangle^+_\Delta .$$



4 FKG INEQUALITY.

Rem. $f : \Omega_\Lambda \to \mathbb{R}$ is incr. if

$$\sigma \leq \sigma' \Rightarrow f(\sigma) \leq f(\sigma').$$

Ex:
- $\sigma \mapsto \sigma_0$ is $\uparrow$
- $\sigma \mapsto \sigma_A$ is <u>not</u> $\uparrow$ if $|A| \geq 2$
- $\forall x \quad n_x : \sigma \mapsto \frac{1+\sigma_x}{2}$ is $\uparrow$
- $\forall A \quad n_A = \underbrace{\prod_{x\in A} n_x}_{\mathbb{1}_{\forall x \in A\ \sigma_x = +1}}$ is $\uparrow$

Prop. $(n_A)_{A\subset\Lambda}$ forms a basis of $\mathbb{R}^\Omega = \{ f : \Omega_\Lambda \to \mathbb{R} \}$

Proof:
- $\# \{ n_A , A \subset \Lambda \} = \dim \mathbb{R}^\Omega$
- $\sigma_B = \prod_{x\in B} (2n_x - 1) \in \mathrm{Vect}(n_A, A\subset\Lambda)$ Hence (n_A) <u>generating</u>.

Thm [FKG inequality]

$$\boxed{\begin{array}{l} \forall f, g : \Omega_\Lambda \longrightarrow \mathbb{R} \quad \text{increasing.} \\ \langle f g \rangle^\omega \geq \langle f \rangle^\omega \langle g \rangle^\omega \end{array}}$$

Proof: $\forall\, \sigma \leq \sigma'$, we have $\dfrac{\mu^\omega(\sigma^{(x)})}{\mu^\omega(\sigma_{(x)})} \leq \dfrac{\mu^\omega(\sigma'^{(x)})}{\mu^\omega(\sigma'_{(x)})}$ ∎

Rk: We say that $f : \Omega_\Lambda \longrightarrow \mathbb{R}$ is decreasing if $-f$ is increasing.

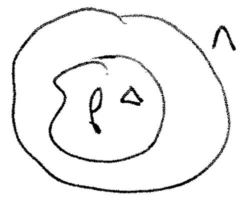
$\forall\, f, g \searrow \quad \langle f g \rangle^\omega \geq \langle f \rangle^\omega \langle g \rangle^\omega$

$\forall\, f \nearrow \; \forall\, g \searrow \quad \langle f g \rangle^\omega \leq \langle f \rangle^\omega \langle g \rangle^\omega$

Application: (Pushing B.C.)

Let $\Delta \subset \Lambda$, let f be an increasing function of $(\sigma_i)_{i \in \Delta}$

Then
$$\langle f \rangle^+_\Delta \geq \langle f \rangle^+_\Lambda$$
$$\langle f \rangle^-_\Delta \leq \langle f \rangle^-_\Lambda$$



Proof: $\langle f \rangle^+_\Delta \overset{\text{DMP}}{=} \dfrac{\langle f \times n_{\partial\Delta} \rangle^+_\Lambda}{\langle n_{\partial\Delta} \rangle^+_\Lambda} \overset{\text{FKG}}{\geq} \langle f \rangle^+_\Lambda$

equivalently for $\langle\ \rangle^-$ replacing $n_{\partial\Delta}$ by the decreasing fct $\mathbb{1}_{\forall x \in \partial\Delta\ \sigma_x = -1}$. ∎

Rk **FKG** : $\forall\, \omega$ B.C. $\langle n_A n_B \rangle^{\omega} \geq \langle n_A \rangle^{\omega} \langle n_B \rangle^{\omega}$.

GKS : $\langle \sigma_A \sigma_B \rangle^{+} \geq \langle \sigma_A \rangle^{+} \langle \sigma_B \rangle^{+}$.

$\langle \sigma_A \sigma_B \rangle^{\emptyset} \geq \langle \sigma_A \rangle^{\emptyset} \langle \sigma_B \rangle^{\emptyset}$

$\rightarrow$ not true for general B.C.