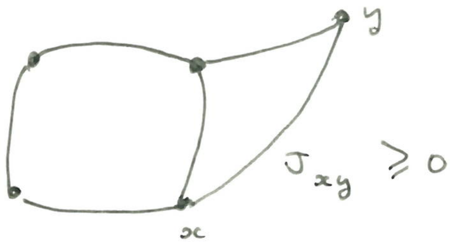
Ising model : CONCLUSION

Finite weighted graph $G = (V, E)$ $J = (J_e)$



$$H(\sigma) = - \sum_{xy \in E} J_{xy} \sigma_x \sigma_y \qquad \mu[\sigma] = \frac{1}{Z} e^{-H(\sigma)}.$$

Theo: Random current representation and switching lemma.

- $$\langle \sigma_A \rangle = \frac{\mathbb{P}[\partial N = A]}{\mathbb{P}[\partial N = \emptyset]}$$

- $$\langle \sigma_A \rangle^2 = \mathbb{P}[M + N \in \mathcal{F}_A \mid \partial M = \partial N = \emptyset]$$

→ "path" interpretation of $\langle \sigma_x \sigma_y \rangle$

→ Exact computation

→ prove inequalities.

- Inequalities

$\boxed{\text{GKS } 1+2}$: $\langle \sigma_A \rangle \geq 0$, $\langle \sigma_A \sigma_B \rangle \geq \langle \sigma_A \rangle \langle \sigma_B \rangle$

$\boxed{\text{MONOTONICITY}}$ $\quad J \leq J' \Rightarrow \langle \sigma_A \rangle_J \leq \langle \sigma_A \rangle_{J'}$

$\boxed{\text{SIMON LIEB}}$ $\quad \langle \sigma_x \sigma_y \rangle \leq \sum_{z \in \partial^i_{in} S} \langle \sigma_x \sigma_z \rangle_S \langle \sigma_z \sigma_y \rangle$

also GMS ...

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$\boxed{\text{Theory 2: STOCHASTIC DOMINATION}}$

μ, ν measures on $\{-1,1\}^V$ $\quad V$ finite set.

- Holley criterion:

$$\left(\forall \, \xi \leq \psi \ \forall x \quad \frac{\mu[\xi^x]}{\mu[\xi_x]} \leq \frac{\nu[\psi^x]}{\nu[\psi_x]} \right) \Rightarrow \mu \ll \nu$$

→ monotonicity properties

$$\langle n_A \rangle^{\omega}_{h} \quad \nearrow \text{ in } h \text{ and } \omega.$$

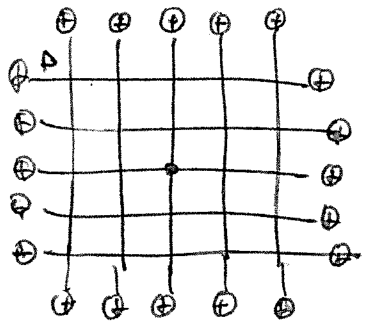
→ correlation inequalities..

$$\langle n_A n_B \rangle \geq \langle n_A \rangle \langle n_B \rangle$$

Rk: $\mu^+_\Lambda,\ \mu^-_\Lambda,\ \mu^0_\Lambda \longrightarrow$ RC + SD.

$$\left.\begin{array}{l}\mu^\omega_\Lambda \quad \omega \text{ general} \\ \mu^+_{\Lambda,h},\ h<0\end{array}\right\} \longrightarrow \text{only SD.}$$

PHASE TRANSITION. $\beta \geq 0$



$$H^+_{\Lambda_n}(\sigma) = -\beta \sum_{\substack{x,y\in\Lambda_n \\ x\sim y}} \sigma_x\sigma_y - \beta \sum_{\substack{x\in\Lambda_n,\, y\in\partial\Lambda_n \\ x\sim y}} \sigma_x$$

$\boxed{P1}$ $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \geq 0$ (GKS 1)

$\boxed{P2}$ $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \searrow$ in n (GKS 2)

$\boxed{P3}$ $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \nearrow$ in β (monotonicity in J)

$$\boxed{m(\beta) = \lim_{n\to\infty} \langle \sigma_0 \rangle^+_{\Lambda_n}}$$

0 $\quad m(\beta)=0 \quad \beta_c \quad m(\beta)>0$

unique infinite volume measure $\qquad \Rightarrow \mu^- \neq \mu^+$.

<u>Properties</u> • $\beta_c(d=1) = +\infty$ (RC)

- $\beta_c > 0$ (Simon Lieb)
- $\beta_c < \infty$ $d\geq 2$ (Peierls)
- Sharpness (Simon Lieb + diff. ineq.).
- uniqueness of the infinite volume measure for $\beta < \beta_c$

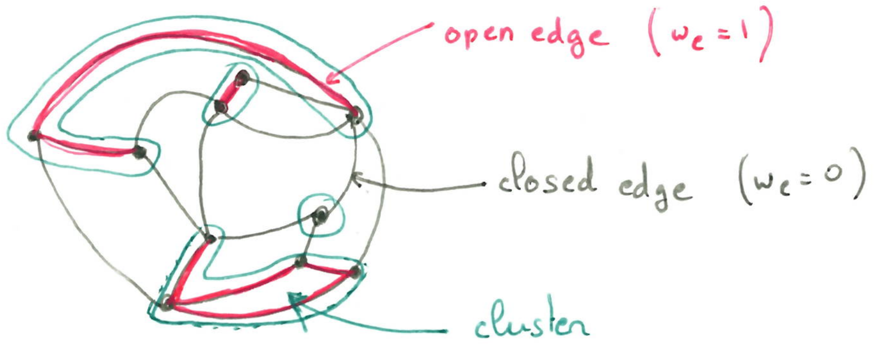
CHAPTER 9

FK- PERCOLATION

$G = (V, E)$ finite graph $\quad p \in [0,1]$ "edge weight"

1 FK- PERCOLATION ON A FINITE GRAPH

Percolation configuration: $\omega = (\omega_e)_{e \in E} \in \{0,1\}^E$.



Rk: $\{\text{percolation config.}\} \overset{\text{bij}}{\simeq} \{\text{subgraphs of } G\}$

$$\omega \longmapsto (V, \{e : \omega_e = 1\})$$

Terminology:
- e is open in ω if $\omega_e = 1$
- e is closed in ω if $\omega_e = 0$
- cluster in ω = connected component of ω.
- open path in ω = path made of open edges.

Not:
- $|\omega| := \sum_{e \in E} \omega_e$ "number of open edges". (above $|\omega| = 7$)
- $|E \setminus \omega| = |E| - |\omega|$ "number of closed edges" (above $|E\setminus\omega| = 10$)
- $k(\omega)$ = "number of clusters in ω" ($k(\omega) = 4$)
- $A \overset{\omega}{\longleftrightarrow} B$: "$\exists$ open path in ω from A to B".

Def: The FK-Ising measure on G with edge weight p is defined by

$$\forall \omega \in \{0,1\}^E \qquad \phi_p[\omega] = \frac{1}{Z} p^{|\omega|} (1-p)^{|E \setminus \omega|} 2^{k(\omega)}$$

$$Z = \sum_{\omega \in \{0,1\}^E} p^{|\omega|} (1-p)^{|E\setminus \omega|} 2^{k(\omega)}.$$

Rk: • $\phi_p[\omega] = \frac{(1-p)^{|E|}}{Z} \left(\frac{p}{1-p}\right)^{|\omega|} 2^{k(\omega)} \underset{\uparrow}{=} \frac{1}{\tilde{Z}} e^{-\beta|\omega|} 2^{k(\omega)}$

$$\boxed{p = 1 - e^{-2\beta}}$$

• Bernoulli perco : $\phi_p(\omega) = p^{|\omega|} (1-p)^{|E \setminus \omega|}$

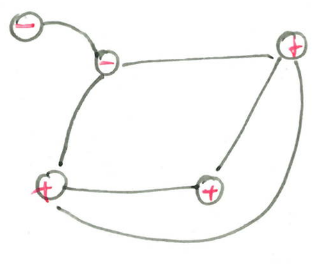
• general FK : $\phi_{p,q}(\omega) = \frac{1}{Z} p^{|\omega|} (1-p)^{|E\setminus\omega|} q^{k(\omega)}$

$q = 1 \to$ Bernoulli

$q = 2 \to$ FK-Ising

2. EDWARDS-SOKAL COUPLING.

$\boxed{p = 1 - e^{-2\beta}}$ $\beta > 0$



Ising on G, β $\quad H(\sigma) = -\beta \sum_{xy \in E} \sigma_x \sigma_y$

$$\mu_\beta[\sigma] = \frac{1}{Z} e^{-H(\sigma)}$$

FK-Ising:

$$\phi(\omega) = \frac{1}{Z} p^{|\omega|} (1-p)^{|E\setminus\omega|} 2^{k(\omega)}.$$

Not: $\sigma \sim \omega$ if σ is constant on the clusters of ω.

$$\left(x \overset{\omega}{\longleftrightarrow} y \Rightarrow \sigma_x = \sigma_y \right)$$

"σ compatible with ω"

Prop. Let $\beta \geq 0$. $p = 1 - e^{-2\beta}$.

The measure P on $\{0,1\}^E \times \{-1,1\}^V$ def. by

$$P[(\omega,\sigma)] = \frac{1}{Z} p^{|\omega|} (1-p)^{|E \setminus \omega|} \mathbb{1}_{\sigma \sim \omega}$$

is a coupling of ϕ_p and μ_β.

$$\left(\text{i.e. } P[\{\omega\} \times \{-1,1\}^V] = \phi_p[\omega] \right.$$
$$\left. P[\{0,1\}^E \times \{\sigma\}] = \mu_\beta[\sigma] \right)$$

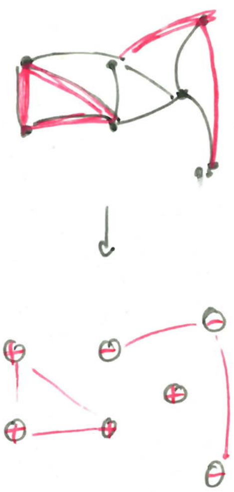
in other words if $(X,Y) \sim P$ then $X \sim \phi_p$ and $Y \sim \mu_\beta$.

Furthermore:
$$\boxed{\mathbb{P}[Y = \sigma \mid X = \omega] = \frac{P[(\omega,\sigma)]}{\phi(\omega)} = \mathbb{1}_{\sigma \sim \omega} \cdot \frac{1}{2^{k(\omega)}}}$$

Recipe to obtain a Ising configuration

① Sample a FK-Ising configuration

② Color indep. each cluster $+1$ or -1 with proba $\frac{1}{2} / \frac{1}{2}$.



Proof: $\cdot \displaystyle\sum_{\sigma} P[(\omega,\sigma)] = \frac{1}{Z} p^{|\omega|}(1-p)^{|E\setminus\omega|} \underbrace{\sum_{\sigma} \mathbb{1}_{\sigma\sim\omega}}_{2^{k(\omega)}}$

$\cdot$ For $\sigma \in \{-1,+1\}^V$ set $A_\sigma = \{xy \in E \ \sigma_x = \sigma_y\}$.

$$\sum_{\omega\in\{0,1\}^E} P[(\omega,\sigma)] = \frac{(1-p)^{|E|}}{Z} \cdot \underbrace{\sum_{\omega\in\{0,1\}^{A_\sigma}} \left(\frac{p}{1-p}\right)^{|\omega|}}_{=\prod_{e\in A_\sigma}\left(1+\frac{p}{1-p}\right)}$$

$$= \frac{(1-p)^{|E|}}{Z} \frac{1}{(1-p)}^{|A_\sigma|}$$

$$= \frac{e^{-\beta|E|}}{Z} e^{+\beta|A_\sigma|}$$

$$= \frac{1}{Z'} e^{-\beta H(\sigma)}. \qquad \blacksquare$$

Application: $\langle \sigma_x \sigma_y \rangle = \phi[x \overset{\omega}{\longleftrightarrow} y]$

Pf: Let $(X,Y) \sim P$

$$\langle \sigma_x\sigma_y\rangle = \underset{\substack{\uparrow \\ \text{coupling}}}{E}[\sigma_x\sigma_y] = \underbrace{E[\sigma_x\sigma_y \mid x \overset{\omega}{\longleftrightarrow} y]}_{=1} P(x \overset{\omega}{\longleftrightarrow} y)$$
$$+ \underbrace{E[\sigma_x\sigma_y \mid x \not\longleftrightarrow y]}_{=0} P(x \overset{\omega}{\not\longleftrightarrow} y)$$
$$= P[x \overset{\omega}{\longleftrightarrow} y].$$

Thm: $\forall A \subset V$

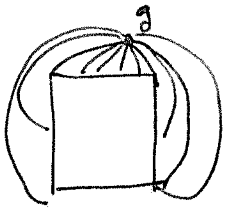
$$\boxed{\langle \sigma_A \rangle = \phi[\mathcal{F}_A]}$$

Pf: escercise.

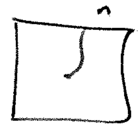
3 PHASE TRANSITION

Prop. $\boxed{\forall p \leq p' \quad \phi_p \ll \phi_{p'}}$

Pf: Holley criterion

$G_n =$  $\quad V = \{-n,\dots,n\}^d \cup \{g\}$

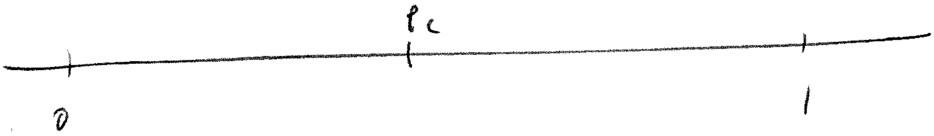
$$\theta(p) = \lim_{n\to\infty} \phi_{p,G_n}[0 \longleftrightarrow g] = \lim_{n\to\infty} \phi_{p,G_n}[\ \ ]$$



Rk 1 $\phi_p^w := \lim \phi_{p,G_n}$ (weak limit well def.)

$$\theta(p) = \phi_p^w[0 \longleftrightarrow \infty]$$

Def: $p_c = \sup\{p : \theta(p) = 0\}$



Rk: $p_c = 1 - e^{-\beta_c} \qquad \theta(p) = m(\beta)$

Thm: β_c in dimension $d=2$

$$p_c = \frac{\sqrt{2}}{1+\sqrt{2}}$$

$$\Leftrightarrow \boxed{\beta_c = \log(1+\sqrt{2}).}$$