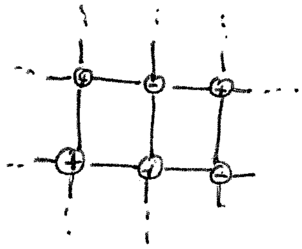
CHAPTER 6 :

ISING MODEL IN INFINITE VOLUME

$\Omega = \{+1,-1\}^{\mathbb{Z}^d}$ $\quad \mathcal{F}$ product σ-algebra



$\beta \geq 0$ $\quad J_{xy} = \mathbb{1}_{x \sim y}$ n.n. interaction

<u>Goal</u>: define μ Ising measure on $(\Omega, \mathcal{F})$.

For $\Lambda \subset\subset \mathbb{Z}^d \rightsquigarrow \mu^{\omega}_{\Lambda,\beta}$ in Ω_Λ

<u>idea 1</u> taking weak limits of μ^{w}_{Λ} as $\Lambda \uparrow \mathbb{Z}^d$

"this is the way we will define μ^+, μ^-"

<u>idea 2</u> via specification (Gibbs formalism).

Call μ an Ising measure on $\mathbb{Z}^d$ if its marginals in finite boxes (when we condition to the configuration outside the box) coincide with the finite volume Ising measures.

1 PRELIMINARIES

In order to construct infinite volume measures, we "need" an extension theorem from measure theory. Since we are working with a product space $\Omega = \{+1,-1\}^{\mathbb{Z}^d}$ we use Kolmogorov's extension theorem. (other approaches can be used, e.g. Riesz theorem, see [Velenik])

Not. $\mathcal{F}_\Lambda = \sigma\left((\sigma_i)_{i\in\Lambda} \right)$.

Rk: $\mathcal{F} = \sigma\left(\bigcup_{\Lambda \subset\subset \mathbb{Z}^d} \mathcal{F}_\Lambda \right)$

Thm: [Kolmogorov extension's theorem]

Consider a function $\mu : \bigcup_{\Lambda\subset\subset \mathbb{Z}^d} \mathcal{F}_\Lambda \longrightarrow \mathbb{R}_+$ s.t.

$\forall \Lambda \subset\subset \mathbb{Z}^d \quad \mu_{|\mathcal{F}_\Lambda}$ is a probability measure on $(\Omega, \mathcal{F}_\Lambda)$.

Then there exists a unique probability measure $\overline{\mu}$ on $\mathcal{F}$ that coincides with μ on every $\mathcal{F}_\Lambda$, $\Lambda \subset\subset \mathbb{Z}^d$.

Ref: The version above is taken from Villani's lecture notes (available at cedricvillani.org / for-mathematicians / lecture-notes [section III.6.5], in French)

Def: A function $f : \Omega \to \mathbb{R}$ is said to be local if there exists $\Lambda \subset\subset \mathbb{Z}^d$ s.t. f is $\mathcal{F}_\Lambda$-measurable.

Rk: The set of local functions is a vector space generated by $(n_A)_{A\subset\subset\mathbb{Z}^d}$.

Rk: If f is $\mathcal{F}_\Lambda$-measurable, then $f = f((\sigma_x)_{x\in\Lambda})$, and therefore f can be seen as a function $f : \Omega_\Lambda \to \mathbb{R}$

$\hookrightarrow \langle f \rangle_\Lambda^\omega$ is well defined.

2. THE INFINITE VOLUME MEASURES μ^+ AND μ^-.

Not: $\Lambda_k \uparrow \mathbb{Z}^d$ if $\Lambda_k \subset \Lambda_{k+1}$ and $\mathbb{Z}^d = \bigcup_{k \geq 1} \Lambda_k$

Thm: "[$\beta \geq 0$]" There exist two probability measures μ^- and μ^+ on $(\Omega, \mathcal{F})$ characterized by $\forall f$ local function $\forall \Lambda_n \uparrow \mathbb{Z}^d$

(i) $\langle f \rangle^+ = \lim_{k \to \infty} \langle f \rangle^+_{\Lambda_k}$

(ii) $\langle f \rangle^- = \lim_{k \to \infty} \langle f \rangle^-_{\Lambda_k}$

where $\langle f \rangle^\omega = \int_\Omega f \, d\mu^\omega$ for $\omega \in \{+,-\}$

Rk: ① If f local, then $\langle f \rangle^+_{\Lambda_k}$ is well defined for k large.

② In (i) and (ii) the limit does not depend on the chosen sequence $\Lambda_k \uparrow \mathbb{Z}^d$.

Proof: We only prove the existence of μ^+ satisfying (i). The uniqueness is a direct consequence of Kolmogorov's theorem. The proof for μ^- is the same

Step 1: construction of μ^+.

Write $B_k = \{-k, \dots, k\}^d$. Let $A \subset\subset \mathbb{Z}^d$. Since n_A is increasing and local, the sequence $\left(\langle n_A \rangle^+_{B_k}\right)_{k \geq k_0}$ is well defined (provided k_0 large) and non increasing;

We can define the decreasing limit:

$$\langle n_A \rangle^+ \underset{\text{def}}{:=} \lim_{k \to \infty} \langle n_A \rangle^+_{B_k}.$$

Now let $\Lambda \subset\subset \mathbb{Z}^d$ and $E \in \mathcal{F}_\Lambda$. Since $(n_A)_{A\subset\Lambda}$ is a basis of $\mathbb{R}^{\Omega_\Lambda}$, we can write $\mathbb{1}_E$ as a linear combination

$$\mathbb{1}_E = \sum_{A\subset\Lambda} \lambda_A \, n_A \qquad \text{for } \lambda_A \in \mathbb{R}.$$

Then we define

$$\mu^+[E] \overset{\text{def}}{:=} \sum_{A\subset\Lambda} \lambda_A \langle n_A \rangle^+.$$

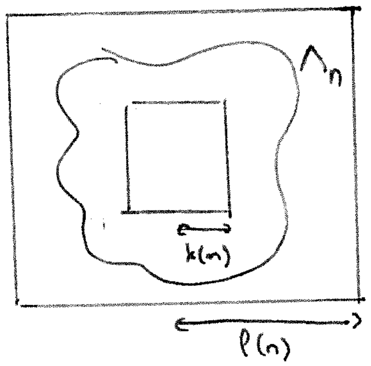
This way, we have defined $\mu^+ : \bigcup_{\Lambda\subset\subset\mathbb{Z}^d} \mathcal{F}_\Lambda \longrightarrow \mathbb{R}_+$, and $\forall \Lambda \subset\subset \mathbb{Z}^d$ $\mu|_{\mathcal{F}_\Lambda}$ probability measure (exercise).

By Kolmogorov's extension theorem, μ^+ can be extended into a probability measure on $\mathcal{F}$.

<u>Step. 2</u> Proof of (i). Let $\Lambda_n \uparrow \mathbb{Z}^d$. Let $(k(n))$ and $(\ell(n))$ be two sequences such that

$$\forall n \quad B_{k(n)} \subset \Lambda_n \subset B_{\ell(n)},$$

and $k(n), \ell(n) \xrightarrow[n\to\infty]{} \infty$.



By monontonicity, we have for every $A \subset\subset \mathbb{Z}^d$ and n large

$$\langle n_A \rangle^+_{B_{k(n)}} \geq \langle n_A \rangle^+_{\Lambda_n} \geq \langle n_A \rangle^+_{B_{\ell(n)}}$$

Hence $\lim_{n\to\infty} \langle n_A \rangle^+_{\Lambda_n} = \langle n_A \rangle^+$, which implies the result (since any local function can be written as a linear combination of the n_A's). ■

<u>To remember</u>: " μ^+ is the decreasing limit of $(\mu^+_{\Lambda_n})_n$ "

in the sense $\langle f \rangle^+ = \lim\downarrow_{n\to\infty} \langle f \rangle^+_{\Lambda_n}$ $\forall f$ local ?.

<u>Rk</u>: in particular $\langle \sigma_0 \rangle^+ = \lim_{n\to\infty} \langle \sigma_0 \rangle^+_{B_n} = m(\beta)$

<u>Prop</u>:

. [Spin flip] $\sigma \sim \mu^+ \iff (-\sigma) \sim \mu^-$

[translation invariance] $\sigma \sim \mu^+ \Rightarrow \forall z \in \mathbb{Z}^d \; \tau_z \sigma \sim \mu^+$.

$\sigma \sim \mu^- \Rightarrow \forall z \in \mathbb{Z}^d \; \tau_z \sigma \sim \mu^-$

where $(\tau_z \sigma)_x = \sigma_{x-z}$.

<u>Proof</u>: exercise : use that $\forall \sigma \in \Omega_{\Lambda_n}$

- $\mu^-_{\Lambda_n}[-\sigma] = \mu^+_{\Lambda_n}[+\sigma]$ and
- $\mu^+_{z+\Lambda_n}[\tau_z \sigma] = \mu^+_{\Lambda_n}[\sigma]$

<u>Consequence</u>.

. $\langle \sigma_0 \rangle^+ = - \langle \sigma_0 \rangle^-$

. $\forall x \; \langle \sigma_x \rangle^+ = \langle \sigma_0 \rangle^+ = m(\beta)$

3 GENERAL INFINITE-VOLUME ISING MEASURE

<u>Not</u>: For $S \subset \mathbb{Z}^d$ (not necessarily finite), we write $\mathcal{F}_S = \sigma((\sigma_x)_{x \in S})$.

<u>Def</u>: A measure on $(\Omega, \mathcal{F})$ is called an infinite-volume Ising measure (or Gibbs state) at inverse temperature β if for every $\Lambda \subset\subset \mathbb{Z}^d$ and f $\mathcal{F}_\Lambda$-measurable

$$\boxed{\langle f \mid \mathcal{F}_{\Lambda^c} \rangle (\sigma) = \langle f \rangle_\Lambda^{\sigma \mid \partial\Lambda} \quad \text{for } \mu\text{-a.e. } \sigma \in \Omega.}$$

conditional expectation of f w.r.t μ.

$\sigma \leftarrow$ sampled according to μ

the law of the configuration inside Λ, conditioned to the configuration outside Λ is $\mu_\Lambda^{\sigma \mid \partial\Lambda}$.

<u>Thm</u>: $\boxed{\mu^+ \text{ and } \mu^- \text{ are infinite-volume Ising measures}}$

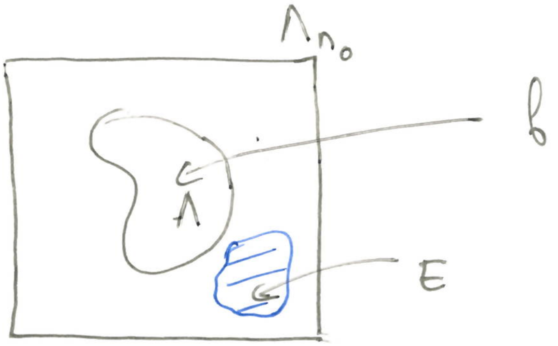
<u>Proof</u>: Let $\Lambda \subset\subset \mathbb{Z}^d$. Let f be a $\mathcal{F}_\Lambda$-measurable function.

We need to prove that for every $E \in \mathcal{F}_{\Lambda^c}$,

$$\langle f(\sigma) \cdot \mathbb{1}_E(\sigma) \rangle^+ = \langle \langle f \rangle_\Lambda^{\sigma \mid \partial\Lambda} \cdot \mathbb{1}_E(\sigma) \rangle^+$$

Since $E \in \mathcal{F}_{\Lambda^c}$ can be approximated by local events ($\forall \varepsilon > 0 \; \exists E_{loc}$ local s.t. $\mu^+[E \triangle E_{loc}] < \varepsilon$), it suffices to prove the equation above for $E \in \mathcal{F}_{\Lambda^c}$ local.

Fix $E \in \mathcal{F}_{\Lambda^c}$ local. Let $\Lambda_n \uparrow \mathbb{Z}^d$. Let n_0 large enough s.t. $\Lambda \subset \Lambda_{n_0}$ and E is $\mathcal{F}_{\Lambda_{n_0}}$-measurable



By the domain Markov property, for every $n \geq n_0$

$$\forall \sigma \in \Omega_{\Lambda_n} \quad \langle f \mid \mathcal{F}_{\Lambda_n \setminus \Lambda} \rangle^+_{\Lambda_n}(\sigma) = \langle f \rangle^{\sigma \mid \partial\Lambda}_{\Lambda}.$$

Therefore

$$\begin{aligned}
\langle f \mathbb{1}_E \rangle^+_{\Lambda_n} &= \langle \langle f \mathbb{1}_E \mid \mathcal{F}_{\Lambda_n \setminus \Lambda} \rangle^+_{\Lambda_n} \rangle^+_{\Lambda_n} \\
&= \langle \langle f \mid \mathcal{F}_{\Lambda_n \setminus \Lambda} \rangle^+_{\Lambda_n} \mathbb{1}_E \rangle^+_{\Lambda_n} \\
&= \langle \langle f \rangle^{\sigma \mid \partial\Lambda}_{\Lambda} \mathbb{1}_E \rangle^+_{\Lambda_n}
\end{aligned}$$

Since f, $\mathbb{1}_E$ and $\sigma \longmapsto \langle f \rangle^{\sigma \mid \partial\Lambda}_{\Lambda}$ are local, we can take the limit ~~as~~ n tends to infinity in the equation above, which concludes the proof. ∎

<u>Prop:</u> Let μ be an infinite-volume Ising measure.
Then for every local fction f $\uparrow$,

$$\langle f \rangle^- \leq \underline{\langle f \rangle} \leq \langle f \rangle^+.$$

"expectation of f w.r.t μ"

<u>Proof:</u> Let $\Lambda_n \uparrow \mathbb{Z}^d$. For n large enough, and $\forall\, \omega \in \{-1,1\}^{\partial\Lambda}$

$$\langle f \rangle^-_{\Lambda_n} \leq \langle f \rangle^\omega_{\Lambda_n} \leq \langle f \rangle^+_{\Lambda_n}$$

Therefore, we have

$$\langle f \rangle^-_{\Lambda_n} \leq \langle f \mid \mathcal{F}_{\Lambda_n^c} \rangle \leq \langle f \rangle^+_{\Lambda_n} \quad \mu\text{-a.s.}$$

Taking the expectation w.r.t. μ and letting n tend to infinity concludes the proof. ■

4 UNIQUENESS

Thm [Characterization of uniqueness]

For fixed $\beta \geq 0$, the following are equivalent.

(i) there exists a unique infinite-Volume Ising measure

(ii) $\mu^- = \mu^+$.

(iii) $m(\beta) = 0$

<u>Proof</u>: (ii) $\Leftrightarrow$ (i) follows from the proposition above.

(ii) $\Rightarrow$ (iii) $\quad \mu^+ = \mu^- \Rightarrow \langle \sigma_0 \rangle^+ = \underbrace{\langle \sigma_0 \rangle^-}_{= -\langle \sigma_0 \rangle^+} \Rightarrow \langle \sigma_0 \rangle^+ = 0$

(iii) $\Rightarrow$ (ii) Let $A \subset\subset \mathbb{Z}^d$. Since the function

$$\sum_{x \in A} n_x - n_A$$

is increasing, we have

$$\Big\langle \sum_{x \in A} n_x - n_A \Big\rangle^- \leq \Big\langle \sum_{x \in A} n_x - n_A \Big\rangle^+ .$$

Therefore,

$$\langle n_A \rangle^+ - \langle n_A \rangle^- \leq \sum_{x \in A} \langle n_x \rangle^+ - \langle n_x \rangle^-$$

$$\underset{\substack{\uparrow \\ \text{translation invariance}}}{=} \frac{|A|}{2} \left(\langle \sigma_0 \rangle^+ - \langle \sigma_0 \rangle^- \right).$$

If $\langle \sigma_0 \rangle^- = \langle \sigma_0 \rangle^+$ then $\forall A \subset\subset \mathbb{Z}^d$ $\langle n_A \rangle^- = \langle n_A \rangle^+$, which implie $\mu^- = \mu^+$ $\blacksquare$

<u>Conclusion:</u>