# 5 MAGNETIZATION AT LOW TEMPERATURE

Theorem $\boxed{\beta_c(2) < \infty}$

Cor: $\boxed{\beta_c(d) < \infty \quad \forall d \geq 2}$

sketch of proof: (Peierls argument)

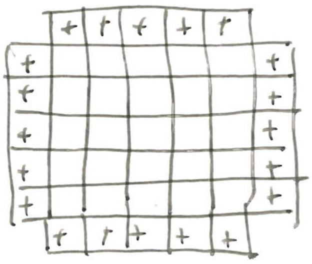
$$\langle \sigma_0 \rangle^+_{\Lambda_n} = \mu^+_{\Lambda_n}[\sigma_0 = 1] - \mu^+_{\Lambda_n}[\sigma_0 = -1]$$

$$= 2\left(\frac{1}{2} - \mu^+_{\Lambda_n}[\sigma_0 = -1]\right)$$

Goal: We prove that $\exists \beta$ large enough s.t.

$$\boxed{\forall n \quad \mu^+_{\Lambda_n,\beta}[\sigma_0 = -1] \leq \frac{1}{4}}$$

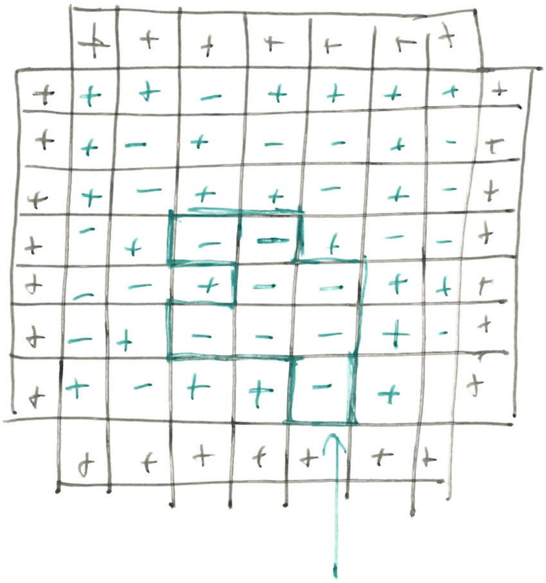
"board" representation of $\Lambda_n \cup \partial\Lambda_n$



$\mu = \mu^+_{\Lambda_n,\beta}$

## Idea 1 :

$\sigma_0 = -1 \Rightarrow \exists$ "disagreement loop" surrounding $0$



$\gamma$ disagreement loop.

## Idea 2 'disagreement loops are costly'

length of $\gamma$

$$\boxed{\mu^+[\underbrace{\gamma \text{ is a disagreement loop}}_{E_\gamma}] \leq e^{-\beta|\gamma|}}$$

↳ def. $\tilde{\sigma}_x := \begin{cases} -\sigma_x & \text{if } x \in \mathrm{Int}(\gamma) \\ +\sigma_x & \text{otherwise} \end{cases}$

$\sigma \to \tilde{\sigma}$ bij

If $\sigma \in E_\gamma$ $\quad H(\tilde{\sigma}) = H(\sigma) - \beta|\gamma|$

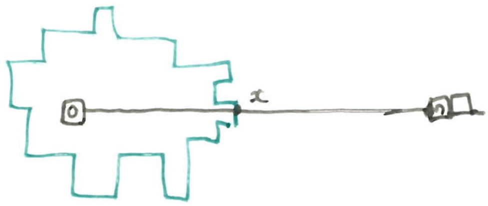
$$\sum_{\sigma \in E_\gamma} e^{-H(\sigma)} = \sum_{\sigma \in E_\gamma} e^{-H(\tilde{\sigma}) - \beta|\gamma|}$$

$$= e^{-\beta|\gamma|} \underbrace{\sum_{\sigma \in E_\gamma} e^{-H(\tilde{\sigma})}}_{\leq Z}$$

$$\mu^{+}[E_\gamma] = \frac{\sum_{\sigma \in E_\gamma} e^{-H(\sigma)}}{Z} \leq e^{-\beta|\gamma|}$$

## Idea 3 : Not too many loops ...

$\Gamma_k := \{$ loop $\gamma$ surrounding $0$ $|\gamma| = k \}$.




$\gamma \in \Gamma_k \Rightarrow \exists x \in \{\frac{1}{2}, \frac{3}{2}, \dots, k - \frac{1}{2}\}$ s.t. $x \in \gamma$

$$|\Gamma_k| \leq \underset{\substack{\uparrow \\ \text{choice of } x}}{k} \times 3^k$$

choices of the steps

indep. of n

Conclusion:

$$\mu[\sigma_0 = -1] \overset{①}{=} \mu\Big[\bigcup_{k \geq 4} \bigcup_{\gamma \in \Gamma_k} E_\gamma\Big]$$

$$\leq \sum_{k \geq 4} \sum_{\gamma \in \Gamma_k} \mu[E_\gamma]$$

$$\overset{②}{\leq} \sum_{k \geq 4} |\Gamma_k| \, e^{-\beta k}$$

$$\overset{③}{\leq} \sum_{k \geq 4} k \times (3e^{-\beta})^k$$

$$\leq \frac{1}{4} \quad \text{if } \beta \geq \underbrace{\beta_1}_{\text{indep. of } n}$$

$\boxtimes$

# CHAPTER 5

## — SHARPNESS —

<u>Framework:</u> $d \geq 1$

- $\mathbb{Z}^d$ $\quad x \sim y$ if $\|x-y\|_1 = 1$

  $\Lambda_n = \{-n, \dots, n\}^d$

- Ising with $+$ BC and n.n interactions

$$H^+_{\Lambda_n,\beta}(\sigma) = -\beta \sum_{\substack{x y \subset \Lambda_n \\ x \sim y}} \sigma_x \sigma_y - \beta \sum_{\substack{x \in \Lambda_n\ y \in \partial \Lambda_n \\ x \sim y}} \sigma_x$$


$$\underbrace{\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \xrightarrow[n\to\infty]{} 0}_{\text{At which speed?}} \qquad \langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \xrightarrow[n\to\infty]{} m(\beta) > 0$$

<u>Goal:</u>
- Introduce $\phi_\beta(S)$.
- Use differential inequalities to obtain quantitative estimate.

**Thm:** [AIZENMAN - BARSKY - FERNANDEZ '87]

Let $\beta < \beta_c$. There exists $c > 0$ s.t.

$$\boxed{\forall n \geq 1 \qquad \langle \sigma_0 \rangle^{+}_{\Lambda_n, \beta} \leq e^{-cn}.}$$

## 1 The quantity $\phi_\beta(S)$

**Def:** Let $S \subset \mathbb{Z}^d$, $\beta \geq 0$.

$$\phi_\beta(S) := \begin{cases} \sum_{x \in \partial_{in} S} \langle \sigma_0 \sigma_x \rangle_S & \text{if } 0 \in S \\ 0 & \text{otherwise} \end{cases}$$

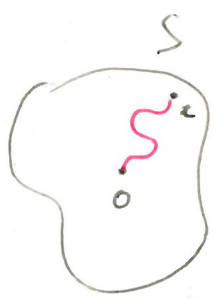
**Rk:** $\langle \cdot \rangle_S$ is Ising on the finite graph

$$G = (S, \{xy \subset S : x \sim y\})$$

(No ghost → "free b.c.")

**Current interpretation:** $N^S$ PPP $(\beta)$ on the graph induced by $S$.

$$\phi_\beta(S) = \frac{1}{\mathbb{P}[\partial N^S = \emptyset]} \sum_{x \in \partial^{in} S} \mathbb{P}[\partial N^S = \{0\} \Delta \{x\}]$$



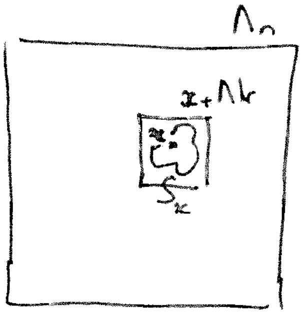

A current with $\partial N^S = 0x$

Prop 1: Let $\beta \geq 0$. If there exists $S \subset \mathbb{Z}^d$, $0 \in S$ s.t. $\phi_\beta(S) < 1$, then $\exists c > 0$ s.t.

$$\forall n \geq 1 \quad \langle \sigma_0 \rangle^+_{\Lambda_n, \beta} \leq e^{-cn}.$$

Proof: Fix $k \geq 1$ s.t. $S \subset \Lambda_k$.

Let $n \geq k$. Let $x \in \Lambda_n$ s.t. $x + \Lambda_k \subset \Lambda_n$




Simon's inequality $\quad S_x = x + S$

$$\langle \sigma_x \rangle^+_{\Lambda_n} \leq \sum_{y \in \partial_{in} S_x} \langle \sigma_x \sigma_y \rangle_{S_x} \langle \sigma_y \rangle^+_{\Lambda_n}$$

$$\leq \phi_\beta(S) \cdot \max_{y \in x + \Lambda_k} \langle \sigma_y \rangle^+_{\Lambda_n}.$$

By induction, we get

$$\langle \sigma_0 \rangle^+_{\Lambda_n} \leq \phi_\beta(S)^{\lfloor \frac{n}{k} \rfloor}.$$

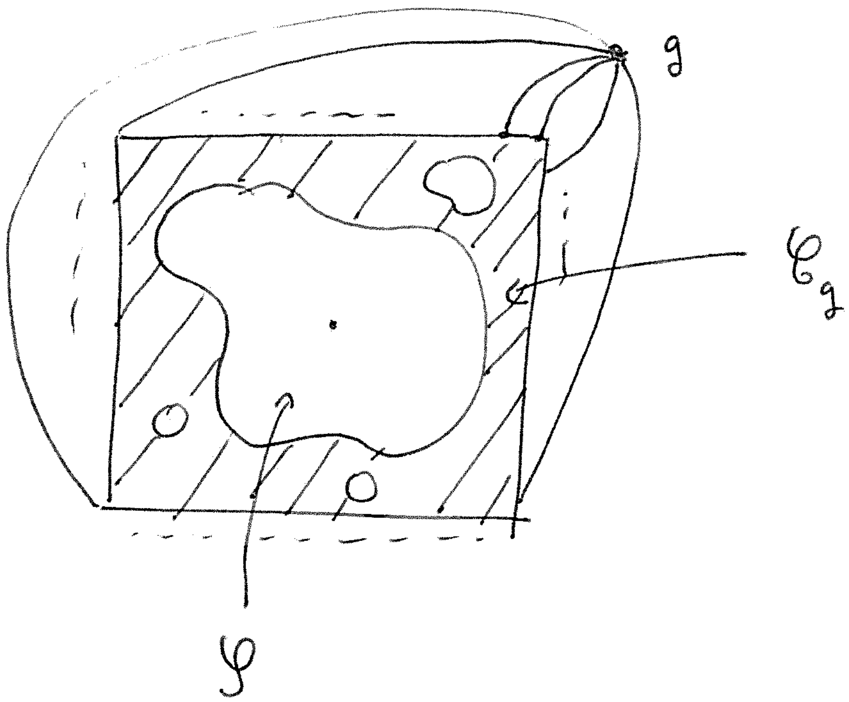
Since $k$ is fixed and $n$ is arbitrary, this concludes the proof. $\blacksquare$

$\underset{(V,E)}{\overset{\shortparallel}{(G}}, J)$ ghost weighted graph corresponding to $\Lambda_n$ with + bc.

Prop. 2 Let $M, N$ be two indep. ppp$(J)$ on $G$.

$$\boxed{\frac{d}{d\beta} \langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \geq \mathbb{E}\left[ \phi_\beta(\mathcal{S}) \,\middle|\, \partial M = \partial N = \emptyset \right]}$$

where $\mathcal{S} = \{ x \in V : x \overset{M+N}{\not\longleftrightarrow} g \}$




Rk: $\mathcal{S}$ is the complement of $\mathcal{C}_g = \{ x \in V : x \overset{M+N}{\longleftrightarrow} g \}$

Pf: see Section 4.