CHAPTER 5

— SHARPNESS —

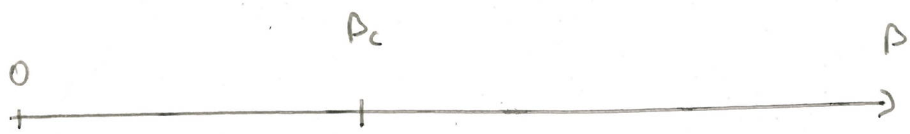
<u>Framework:</u> $d \geq 1$

- $\mathbb{Z}^d$ $\quad x \sim y$ if $\|x-y\|_1 = 1$

  $\Lambda_n = \{-n, \dots, n\}^d$

- Ising with + BC and n.n interactions

$$H^+_{\Lambda_n,\beta}(\sigma) = -\beta \sum_{\substack{x\,y \subset \Lambda_n \\ x\sim y}} \sigma_x \sigma_y - \beta \sum_{\substack{x\in\Lambda_n\ y\in\partial\Lambda_n \\ x\sim y}} \sigma_x$$



$$\underbrace{\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \xrightarrow[n\to\infty]{} 0}_{\text{At which speed?}} \qquad \langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \xrightarrow[n\to\infty]{} m(\beta) > 0$$

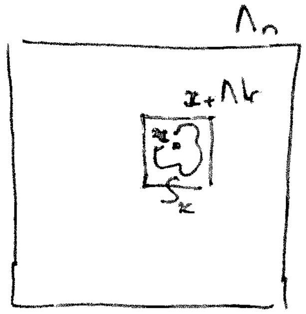
<u>Goal:</u>
- Introduce $\phi_\beta(S)$.
- Use differential inequalities to obtain quantitative estimate.

Prop 1: Let $\beta \geq 0$. If there exists $S \subset\subset \mathbb{Z}^d$, $0 \in S$ s.t. $\phi_\beta(S) < 1$, then $\exists c > 0$ s.t.

$$\forall n \geq 1 \quad \langle \sigma_0 \rangle^+_{\Lambda_n, \beta} \leq e^{-cn}.$$

__Proof__: Fix $k \geq 1$ s.t. $S \subset \Lambda_k$.

Let $n \geq k$. Let $x \in \Lambda_n$ s.t. $x + \Lambda_k \subset \Lambda_n$



Simon's inequality $\quad S_x = x + S$

$$\langle \sigma_x \rangle^+_{\Lambda_n} \leq \sum_{y \in \partial_{in} S_x} \langle \sigma_x \sigma_y \rangle_{S_x} \langle \sigma_y \rangle^+_{\Lambda_n}$$

$$\leq \phi_\beta(S) \cdot \max_{y \in x + \Lambda_k} \langle \sigma_y \rangle^+_{\Lambda_n}$$

By induction, we get

$$\langle \sigma_0 \rangle^+_{\Lambda_n} \leq \phi_\beta(S)^{\lfloor \frac{n}{k} \rfloor}.$$

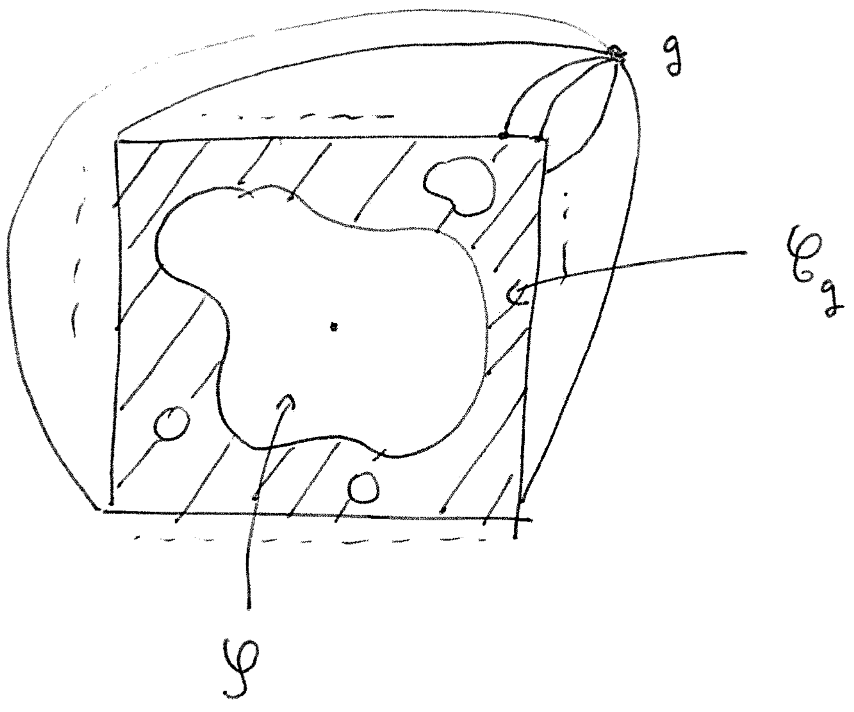
Since k is fixed and n is arbitrary, this concludes the proof. ∎

(G, J) ghost weighted graph corresponding to
$\overset{\shortparallel}{(V,E)}$ Λ_n with $+$ bc.

Prop. 2 Let M, N be two indep. ppp(J) on G.

$$\boxed{\frac{d}{d\beta}\langle\sigma_0\rangle^+_{\Lambda_n,\beta} \geq \mathbb{E}\left[\phi_\beta(\mathscr{S}) \,\middle|\, \partial M = \partial N = \emptyset\right]}$$

where $\mathscr{S} = \{x \in V : x \overset{M+N}{\not\longleftrightarrow} g\}$



Rk: $\mathscr{S}$ is the complement of $\mathscr{C}_g = \{x \in V : x \overset{M+N}{\longleftrightarrow} g\}$

Pf: see Section 4.

2 PROOF OF SHARPNESS.

Alternative critical value:

$$\tilde{\beta}_c : \sup \{ \beta : \exists S \subset\subset \mathbb{Z}^d , 0 \in S , \phi_\beta(S) < 1 \}$$

Rk: $\phi_\beta(S) = \sum_{x \in \partial_{in} S} \langle \sigma_0 \sigma_x \rangle_{S,\beta} \quad \nearrow$ in β.

$\tilde{\beta}_c$

$\exists S : \phi_\beta(S) < 1 \qquad \forall S \quad \phi_\beta(S) \geq 1$

<u>Fact 1</u>: $\forall \beta < \tilde{\beta}_c \quad \exists c > 0$

$$\forall n \geq 1 \quad \langle \sigma_0 \rangle^+_{\Lambda_n} \leq e^{-cn}.$$

$\hookrightarrow$ follows from prop. 1. $\blacksquare$

<u>Fact 2</u>: $\forall \beta \geq \tilde{\beta}_c$

$$m(\beta) \geq \frac{\beta - \tilde{\beta}_c}{1 + \beta - \tilde{\beta}_c}.$$

$\hookrightarrow$ fix $n \geq 1$ and set $f(\beta) := \langle \sigma_0 \rangle^+_{\Lambda_n, \beta}$.

By Prop. 2, $\forall \beta > \tilde{\beta}_c$

$$\begin{aligned} f'(\beta) &\geq \mathbb{E}[\mathbb{1}_{\mathcal{G} \ni 0} \mid \partial M = \partial N = \emptyset] \\ &= \mathbb{P}[0 \leftrightarrow g \mid \partial M = \partial N = \emptyset] \\ &\overset{\text{switch}}{=} 1 - f(\beta)^2 \geq 1 - f(\beta) \end{aligned}$$

Hence $\forall \beta > \tilde{\beta}_c \quad \dfrac{f'}{1-f} \geqslant 1$

Integrating from $\tilde{\beta}_c$ to β, we get

$$\log\left(\frac{1-f(\tilde{\beta}_c)}{1-f(\beta)}\right) \geqslant \beta - \tilde{\beta}_c .$$

$$\overset{x \geqslant \log(1+x)}{\geqslant} \log(1+\beta-\tilde{\beta}_c)$$

Therefore

$$\langle \sigma_0 \rangle^+_{\Lambda_n} = f(\beta) \geqslant \frac{\beta - \tilde{\beta}_c}{1+\beta-\tilde{\beta}_c}$$

The fact follows by letting $n \longrightarrow \infty$. ∎

<u>Conclusion.</u>

$\tilde{\beta}_c$

$$\underbrace{\exists s \quad \phi_\beta(s) < 1}_{} \qquad\qquad \underbrace{\forall s \quad \phi_\beta(s) \geq 1}_{}$$

$\Downarrow$ Prop. 1 $\qquad\qquad$ $\Downarrow$ Prop.

$\langle \sigma_0 \rangle^+_{\Lambda_n} \leq e^{-cn}$ $\qquad\qquad$ $m(\beta) \geq \frac{1}{2}(\beta - \tilde{\beta}_c)$

$(m(\beta) = 0)$ $\qquad\qquad$ $(m(\beta) > 0)$.

The two facts imply that $\beta_c = \tilde{\beta}_c$, which concludes that the exponential decay of $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta}$ also holds for all $\beta < \beta_c$.

3 Derivative formula.

Prop. Let $f : \Omega_{\Lambda_n} \longrightarrow \mathbb{R}$. $\overline{H}(\sigma) = \frac{-1}{\beta} H(\sigma)$.

$$\frac{d}{d\beta} \langle f \rangle_\beta^+ = \langle f \overline{H} \rangle_\beta^+ - \langle f \rangle_\beta^+ \langle \overline{H} \rangle_\beta^+$$

Proof: $Z_\beta^+[f] = \sum_{\sigma \in \Omega_{\Lambda_n}} f(\sigma)\, e^{\beta \overline{H}(\sigma)}$

- $$\frac{\frac{d}{d\beta} Z_\beta^+[f]}{Z_\beta^+} = \frac{1}{Z_\beta^+} \sum_\sigma f(\sigma) \times \overline{H}(\sigma)\, e^{\beta \overline{H}(\sigma)}$$

$$= \langle f \overline{H} \rangle_\beta^+$$

- $$\frac{d}{d\beta} \langle f \rangle_\beta^+ = \underbrace{\frac{\frac{d}{d\beta} Z_\beta^+[f]}{Z_\beta^+}}_{\langle f \overline{H} \rangle} - \underbrace{\frac{Z_\beta^+[f]}{Z_\beta^+}}_{\langle f \rangle} \cdot \underbrace{\frac{\frac{d}{d\beta} Z_\beta^+[1]}{Z_\beta^+}}_{\langle \overline{H} \rangle} .$$

Corollary: Let $\underset{(V,E)}{\overset{}{G}}$, J be the weighted graph (with ghost)

$G = (V,E)$

corresponding to Λ_n with $+$ b.c.

$$\frac{d}{dp}\langle\sigma_0\rangle^+_{\Lambda_n} \geq \sum_{xy\in E} \langle\sigma_0\sigma_x\sigma_y\sigma_g\rangle_G - \langle\sigma_0\sigma_g\rangle_G\langle\sigma_x\sigma_y\rangle_G$$

Proof: $\langle\sigma_0 \bar{H}\rangle^+ - \langle\sigma_0\rangle^+ \langle\bar{H}\rangle^+$

$$= \sum_{\substack{xy\subset\Lambda_n \\ x\sim y}} \langle\sigma_0\sigma_x\sigma_y\rangle^+ - \langle\sigma_0\rangle^+\langle\sigma_{xy}\rangle^+$$

$$+ \sum_{\substack{x\in\Lambda_n\ y\in\partial\Lambda_n \\ x\sim y}} \langle\sigma_0\sigma_x\rangle^+ - \langle\sigma_0\rangle^+\langle\sigma_x\rangle^+$$

$$\geq \sum_{\substack{xy\subset\Lambda_n \\ x\sim y}} \langle\sigma_0\sigma_g\sigma_x\sigma_y\rangle_G - \langle\sigma_0\sigma_g\rangle_G\langle\sigma_x\sigma_y\rangle_G$$

$$+ \sum_{\substack{x\in\Lambda \\ \exists y\in\partial\Lambda\, x\sim y}} \langle\sigma_0\sigma_g\sigma_x\sigma_g\rangle_G - \langle\sigma_0\sigma_g\rangle_G\langle\sigma_x\sigma_g\rangle_G$$

$$= \sum_{xy\in E} \langle\sigma_0\sigma_g\sigma_x\sigma_y\rangle_G - \langle\sigma_0\sigma_g\rangle_G\langle\sigma_x\sigma_y\rangle_G \qquad \blacksquare$$

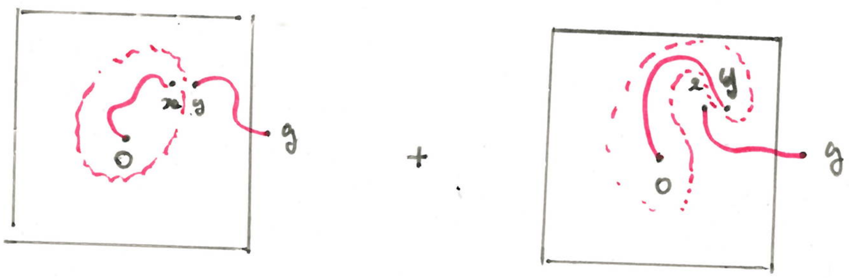
4 PROOF OF PROP. 2

- $$\frac{d}{d\beta} \langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \geq \sum_{xy \in E} \langle \sigma_0 \sigma_g \sigma_x \sigma_y \rangle_G - \langle \sigma_0 \sigma_g \rangle_G \langle \sigma_x \sigma_y \rangle_G$$

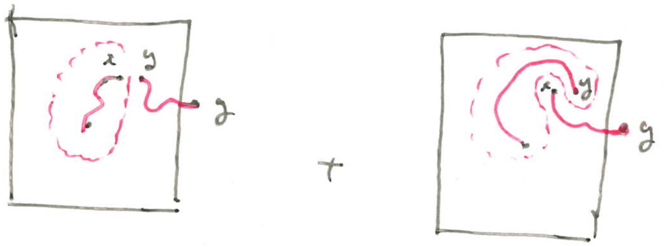
$$\overset{\text{switch.}}{=} \frac{1}{\mathbb{P}[\partial M = \partial N = \emptyset]} \sum_{xy \in E} \mathbb{P}[\partial M = ogxy, \partial N = \emptyset, x \overset{M+N}{\not\leftrightarrow} y]$$

(Convention: $ogxy = \{og\} \Delta \{xy\}$ $\quad ox = \{0\} \Delta \{x\}$.

Diagram. represent. of $\{ \partial M = ogxy, \partial N = \emptyset, x \overset{M+N}{\not\leftrightarrow} y \}$



Diagrammatic representation of $\{ x \overset{n+N}{\longleftrightarrow} y \, , \partial M = ogxy \, , \partial N = \emptyset \}$



$$\mathbb{E}[\, \phi_\beta(S), \partial M = \emptyset, \partial N = \emptyset \,]$$

$$= \sum_{\substack{S \subset V \\ o \in S}} \phi_\beta(S) \, \mathbb{P}[\, \mathcal{S} = S, \partial M = \emptyset, \partial N = \emptyset \,] \qquad (*)$$

<u>Rk:</u> (i) $\mathcal{S} = S$ is meas. wrt. $N - N^S$ and $M - M^S$

(ii) If $\mathcal{S} = S$, all the edges e at the boundary of S (ie $|e \cap S| = 1$) satisfy $M_e = N_e = 0$

Using these two remarks we find

$$\mathbb{P}[\, \mathcal{S} = S, \partial M = \emptyset, \partial N = \emptyset \,]$$

$$\overset{(ii)}{=} \mathbb{P}[\, \mathcal{S} = S, \partial N = \emptyset, \partial (M - M^S) = \emptyset, \partial M^S = \emptyset \,]$$

$$\overset{(i)}{=} \mathbb{P}[\, \mathcal{S} = S, \partial N = \emptyset, \partial M - M^S = \emptyset \,] \, \mathbb{P}[\, \partial M^S = \emptyset \,]$$

Recall $\displaystyle \phi_\beta(S) = \sum_{\substack{xy \in E \\ x \in S \, y \notin S}} \frac{\mathbb{P}[\partial M^S = ox]}{\mathbb{P}[\partial M^S = \emptyset]}$

Plugging the two equations above into (*), we get

$$\mathbb{E}[\phi_\beta(\mathcal{S}), \partial M = \emptyset, \partial N = \emptyset]$$

$$= \sum_{\substack{S \subset V \\ 0 \in S}} \sum_{xy \in E} \mathbb{1}_{\substack{x \in S \\ y \notin S}} \mathbb{P}[\mathcal{S} = S, \partial M = 0x, \partial N = \emptyset]$$

$$= \sum_{xy \in E} \sum_{\substack{S \subset V \\ 0, x \in S \\ y \notin S}} \mathbb{P}[\mathcal{S} = S, \partial M = 0x, \partial N = \emptyset]$$

$$= \sum_{xy \in E} \mathbb{P}[x \overset{M+N}{\not\leftrightarrow} y, y \overset{M+N}{\leftrightarrow} g, \partial M = 0x, \partial N = \emptyset]$$

$$\overset{\text{switching}^*}{=} \sum_{xy \in E} \mathbb{P}[x \overset{M+N}{\not\leftrightarrow} y, \partial M = 0xyg, \partial N = yg]$$

$$\overset{\text{Lemma.}}{\leq} \sum_{xy \in E} \mathbb{P}[x \overset{M+N}{\not\leftrightarrow} y, \partial M = 0xyg, \partial N = \emptyset]$$

$$= \mathbb{P}[\partial M = \partial N = \emptyset] \times \frac{d}{d\beta} \langle \sigma_0 \rangle^+_{\Lambda_n}. \qquad \blacksquare$$

* we use a slight generalization: $\forall E$ event $\mathbb{P}[\partial M = A, \partial N = B, M+N \in \mathcal{F}_C \cap E]$
$= \mathbb{P}[\partial M = A \Delta C, \partial N = B \Delta C, M+N \in \mathcal{F}_C \cap E]$