CHAPTER 6

STOCHASTIC DOMINATION.

Motivation: Ising on $\Lambda \subset\subset \mathbb{Z}^d$ with general b.c.

μ^ω on $\{0,1\}$ ω = spin configuration at $\partial\Lambda$

↳ idea.: compare μ^ω and $\mu^{\omega'}$ when $\omega \leq \omega'$.

Goal: introduce the general concept of stochastic domination

on Ω = general Polish space with partial ordering

$\Omega = \mathbb{R}$

$\Omega = \{-1,1\}^V$ finite set.

Framework: . Ω = Polish space (metric separable, complete) equiped with partial ordering $\leq$

. Borel σ. algebra.

Idea: . the elements of Ω can be compared via $\leq$.

can we compare random elements ?

I DEFINITION AND FIRST EXAMPLES.

Def. Let μ, ν be two proba. measures on Ω.
We say that μ is stoch. dominated by ν (written $\mu \ll \nu$) if:
for all $f : \Omega \to \mathbb{R}$ meas. bounded increasing

$$\int f \, d\mu \leqslant \int f \, d\nu$$

Rk: If $X \sim \mu$ and $Y \sim \nu$. This corresponds to

$$E[f(X)] \leqslant E[f(Y)]$$

Exple 1: $\Omega = \mathbb{R}_+$. For $x > 0$ consider μ_x to be the law of a uniform r.v. on $[0, x]$.
(ie $d\mu_x = \mathbb{1}_{[0,x]} * \frac{1}{x} \, dt$)
We have

$$x \leqslant y \Rightarrow \mu_x \ll \mu_y$$

Pf: Let X be a unif. r.v. on $[0, x]$
Set $Y := \frac{y}{x} \cdot X$ (uniform on $[0, y]$)

We have $X \leq Y$ a.s.

Hence $\forall f \uparrow$ meas. bded $f(X) \leq f(Y)$.

Taking the expectation, we conclude

$$\underbrace{\mathbb{E}[f(X)]}_{\int f \, d\mu_x} \leq \underbrace{\mathbb{E}[f(Y)]}_{\int f \, d\mu_y} .$$

Example 2 $\Omega = \{0,1\}$. For $p \in [0,1]$, set $\mu_p = \text{Bernoulli}(p)$

$$p \leq q \Rightarrow \mu_p \leq \mu_q .$$

Proof. Let U be uniform r.v. in $[0,1]$

Define $X = \begin{cases} 1 & \text{if } U \leq p \\ 0 & \text{if } U > p \end{cases}$, $Y = \begin{cases} 1 & \text{if } U \leq q \\ 0 & \text{if } U > q \end{cases}$.

$p \leq q \Rightarrow X \leq Y$ a.s. $\Rightarrow \forall f : \Omega \to \mathbb{R} \uparrow$ $f(X) \leq f(Y)$

$$\Rightarrow \underbrace{\mathbb{E}[f(X)]}_{\int f \, d\mu_p} \leq \mathbb{E}[f(Y)]$$

$$\int f \, d\mu_p \leq \int f \, d\mu_q . \quad \blacksquare$$

In the two examples above, we prove $\mu \ll \nu$ by using a coupling argument: we construct two r.v. X, Y on the same probability space s.t

$$X \sim \mu \quad Y \sim \nu \qquad X \leq Y \text{ a.s.}$$

The existence of such coupling ensures that $\mu \ll \nu$.
A theorem of Strassen asserts that the reciprocal holds.

Thm. Let Ω be a Polish space, with $\leq$.
Let μ, ν be two proba. measure on Ω
The following are equivalent:

(i) $\mu \ll \nu$ $\rightsquigarrow$ "measure theoretical view point"

(ii) $\exists$ two r.v. $X \sim \mu$ $Y \sim \nu$ on the same proba space s.t.

$$X \leq Y \text{ a.s.}$$ $\rightsquigarrow$ "proba view point"

Pf: admitted. see Lindvall '99 (e.c.p) or Werner [percolation & modèle d'Ising p. 98] for the case Ω finite.

2 STOCHASTIC DOMINATION ON PRODUCT SPACE

V = fixed finite set

$\Omega = \{-1, +1\}^V$, $\leq$ product ordering

$$\cdot\ \eta \leq \psi \iff \forall x \in V \quad \eta(x) \leq \psi(x)$$

Rk: all the section can be adapted to $\Omega = S^V$, S finite ordered

Exercise: $\mu_p := (\text{Bernoulli}(p))^{\otimes V}$ $\quad \left(\mu_p(\eta) = p^{|\eta|}(1-p)^{|V|-|\eta|} \right.$ where $\left. |\eta| = |\{ i : \eta_i = +1 \}| \right)$

$\cdot$ Prove that $p \leq q \Rightarrow \mu_p \ll \mu_q$

Q: How to prove $\mu \ll \nu$ for μ, ν non product measure?

TR: [Holley criterion]

Let μ, ν be two positive measures on Ω.
(ie $\mu(\eta), \nu(\eta) > 0 \ \forall \eta \in \Omega$).

If

$$\forall \eta \leq \psi \qquad \boxed{\frac{\mu[\eta^x]}{\mu[\eta_x]} \leq \frac{\nu[\psi^x]}{\nu[\psi_x]}}$$

Then $\mu \ll \nu$

Not: $\eta^x(y) = \begin{cases} \eta(y) & y \neq x \\ +1 & y = x \end{cases}$ $\qquad \eta_x(y) = \begin{cases} \eta(y) & y \neq x \\ -1 & y = x \end{cases}$.

Glauber dynamics for μ

Notation : for $x \in V$, $\eta \in \Omega$ $\quad \mu_x^\eta := \mu[\,\cdot \mid \forall y \neq x \;\; \omega(y) = \eta(y)]$

$\hookrightarrow$ "resampling at x"

Consider the Markov chain $X = (X_n)_{n \in \mathbb{N}}$ on Ω def. by

- $X_0 \sim \delta_0$
- Assume $X_n = \eta$ for $n \geq 0$

$\rightarrow$ Pick $x \in V$ uniformly at random (indep. of the past)

$\rightarrow$ Define X_{n+1} according to the measure μ_x^η.

Equiv. X_n is a MC with initial distribution δ_0 and transition probability

- $P(\eta, \psi) = \dfrac{1}{|V|} \mu_x^\eta[\psi]$ if η and ψ differ only at x
- $P(\eta, \eta) = \dfrac{1}{|V|} \sum_{x \in V} \mu_x^\eta[\eta]$
- $P(\eta, \psi) = 0$ if η, ψ differ at ≥ 2 vertices.

Prop. X is a irreducible aperiodic MC with invariant distribution μ. In particular $\forall f : \Omega \to \mathbb{R}$

$$\mathbb{E}[f(X_n)] \longrightarrow \int f \, d\mu.$$

Pf: Exercise

Proof of Th (Holley)

Let $\eta \leq \psi$. Let $x \in V$

$$\mu_x^\eta[\omega(x)=1] = \frac{\mu[\eta^x]}{\mu[\eta^x]+\mu[\eta_x]} = \frac{1}{1+\frac{\mu[\eta_x]}{\mu[\eta^x]}}$$

$$\leq \frac{1}{1+\frac{\nu[\psi_x]}{\nu[\psi^x]}} = \nu_x^\psi[\omega(x)=+1].$$

Let $U_1, U_2, \dots$ iid uniform in $[0,1]$.
$Z_1, Z_2, \dots$ iid uniform in V.

We construct two coupled MC. $X=(X_n)_{n\in\mathbb{N}}$ and $Y=(Y_n)_{n\in\mathbb{N}}$, where X is Glauber(μ) and Y is Glauber(ψ) and

$$\forall n \quad X_n \leq Y_n. \quad (*)$$

set $(X_0, Y_0) = (0, 0)$

For $n \geq 0$ define (X_{n+1}, Y_{n+1}) as follows

for $x \neq Z_{n+1}$, set $X_{n+1}(x) = X_n(x)$ $\quad Y_{n+1}(x) = Y_n(x)$

for $x = Z_{n+1}$, set

$$X_{n+1}(x) = \begin{cases} +1 & \text{if } U_{n+1} \leq \mu_x^\eta[\omega(x)=+1] \\ -1 & \text{otherwise} \end{cases} \quad \text{and} \quad Y_{n+1}(x) = \begin{cases} +1 & \text{if } U_{n+1} \leq \nu_x^\eta[\omega(x)=+1] \\ -1 & \text{otherwise.} \end{cases}$$

By induction, (*) holds and therefore, for every $f:\Omega\to\mathbb{R}\uparrow$

$$\mathbb{E}[f(X_n)] \leq \mathbb{E}[f(Y_n)]$$

and we obtain

$$\int f\,d\mu \leq \int f\,d\nu$$

by letting $n\to\infty$. ■

<u>Application</u> If $G=(V,E)$, $J\geq 0$ is a weighted graph $g\in V$ ghost, $\mu=$ Ising $\mu^+=\mu[\cdot\,|\,\sigma_g=+1]$

$$\mu \ll \mu^+.$$

<u>Rk</u>: By considering P the invariant measure of the MC (X_n,Y_n) one obtains a coupling* between μ and ν s.t.

$$P[(\eta,\psi)\in\Omega^2 : \eta\leq\psi]=1.$$

(*ie $P[\cdot\times\Omega)=\mu$ and $P[\cdot\times\Omega]=\nu$)

3 FKG INEQUALITY.

<u>Th</u>: Let μ be a >0 measure on $\Omega=\{-1,1\}^V$. If

$$\forall x\in V\ \forall\eta\leq\psi \quad \frac{\mu(\eta^x)}{\mu(\eta_x)} \leq \frac{\mu(\psi^x)}{\mu(\psi_x)}$$

Then μ satisfies the FKG inequality

$$\boxed{\forall f,g:\Omega\to\mathbb{R}\uparrow \quad \int fg\,d\mu \geq \int f\,d\mu \cdot \int g\,d\mu.}$$

<u>Proof:</u> WLOG $f(\eta) > 0 \ \forall \eta \in \Omega$ (otherwise consider $f + C$, C large constant)

Consider the >0 proba measure ν defined by

$$\forall \psi \in \Omega \quad \nu[\psi] := \frac{1}{\int f \, d\mu} \times f(\psi) \cdot \mu[\psi]$$

Since $f \nearrow$, we have $\forall \eta \leq \psi \ \forall x \in V$

$$\frac{\nu[\psi^x]}{\nu[\psi_x]} = \frac{f(\psi^x)}{f(\psi_x)} \cdot \frac{\mu[\psi^x]}{\mu[\psi_x]} \geqslant \frac{\mu[\eta^x]}{\mu[\eta_x]}$$

Hence by Holley criterion $\mu \ll \nu$. For every $g \nearrow$

$$\int g \, d\mu \leq \int g \, d\nu = \frac{1}{\int f d\mu} \times \int f g \, d\mu . \qquad \blacksquare$$

<u>Application:</u> . FKG for Bernoulli perco $P_p = (\text{Bernoulli}(p))^{\otimes V}$

. FKG for Ising Model (later)