CHAP. 4:

PHASE TRANSITION.

Framework: $d \geq 1$.

- $\mathbb{Z}^d = \{ (x_1, \dots, x_d),\ x_1, \dots, x_d \in \mathbb{Z} \}$ $\quad E(\mathbb{Z}^d) = \{\{x,y\} \subset \mathbb{Z}^d ; \|x-y\|_1 = 1\}$
- $x \sim y$ if $xy \in E(\mathbb{Z}^d)$.

Goals
- define the magnetization $m(\beta)$
- define the critical parameter $\beta_c(d)$
- Show that $0 < \beta_c(d) < \infty$ if $d \geq 2$.

Difficulty: Ising on the infinite graph $\mathbb{Z}^d, E(\mathbb{Z}^d)$

two approaches: $\to$ finite subgraphs of $\mathbb{Z}^d$

$\rightsquigarrow (\mu_\Lambda)_{\Lambda \subset\subset \mathbb{Z}^d}$ [Chap. 4-5]

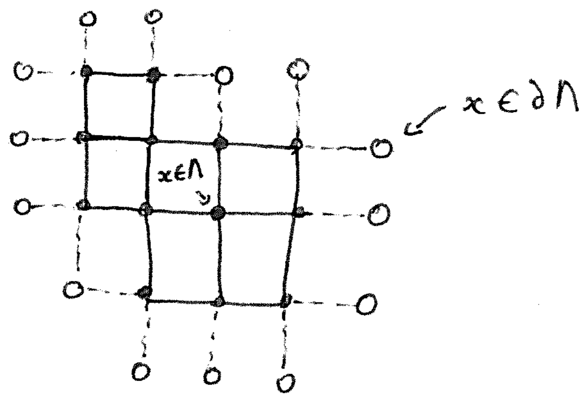
$\to$ define Ising measures on

$\Omega = \{+1, -1\}^{\mathbb{Z}^d}$ [Chap. 6-7.]

1 ISING ON FINITE SUBGRAPHS WITH + B.C.

Let $\Lambda \subset\subset \mathbb{Z}^d$ (i.e $\Lambda \subset \mathbb{Z}^d$ and $|\Lambda| < \infty$)

<u>Boundary:</u> $\partial\Lambda = \{x \in \mathbb{Z}^d \setminus \Lambda : \exists y \in \Lambda \ x \sim y\}$



<u>Spin config.</u>

$$\Omega = \Omega_\Lambda = \{+1, -1\}^\Lambda$$

<u>Hamiltonian</u> Let $\beta \geq 0$. For $\sigma \in \Omega$

$$\boxed{H^+(\sigma) = H^+_{\Lambda,\beta}(\sigma) = -\beta \sum_{\substack{x y \subset \Lambda \\ x \sim y}} \sigma_x \sigma_y - \beta \sum_{\substack{x \in \Lambda, y \in \partial\Lambda \\ x \sim y}} \sigma_x}$$

<u>Ising on Λ with + - b.c.</u>

$$\boxed{\mu^+(\sigma) = \mu^+_{\Lambda,\beta}(\sigma) = \frac{1}{Z} e^{-H(\sigma)}}$$

where $Z^+ = Z^+_{\Lambda,\beta} = \sum_{\sigma \in \Omega} e^{-H^+(\sigma)}$

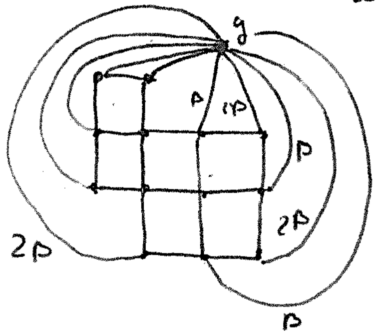
Relation to Ising on a finite graph.

$G = (V, E)$ $\quad$. $V = \Lambda \cup \{g\}$

. $E = \{xy \subset \Lambda : x \sim y\} \cup \{xg ; \exists y \in \mathbb{Z}^d \setminus \Lambda,\ x \sim y\}$

. $J_{xy} = \beta,\ xy \subset \Lambda$

. $J_{xg} = \sum_{\substack{y \notin \Lambda \\ x \sim y}} \beta \ ,\ $ if $x \in \Lambda$.



All the theory from chap. 1-3 applies:

- $\forall A \subset \Lambda \qquad \langle \sigma_A \rangle^+ = \begin{cases} \langle \sigma_A \rangle_G & A \text{ even} \\ \langle \sigma_{A \Delta \{g\}} \rangle_G & A \text{ odd.} \end{cases}$

- GKS ineq: $\forall A, B \subset \Lambda$

$$\langle \sigma_A \rangle^+ \geq 0$$

$$\langle \sigma_A \sigma_B \rangle^+ \geq \langle \sigma_A \sigma_B \rangle^+$$

- Monoton. If $\beta \leq \beta'$

$$\langle \sigma_A \rangle^+_\beta \leq \langle \sigma_A \rangle^+_{\beta'}$$

- Simon-Lieb $\quad S \subset \Lambda \quad x \in S$

$$\langle \sigma_x \rangle^+ \leq \sum_{y \in \partial_{in} S} \langle \sigma_x \sigma_y \rangle_S \ \langle \sigma_y \rangle^+$$

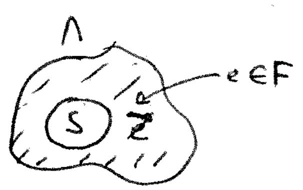
2. Push. BC.

Lem: Let $A \subset S \subset \Lambda \subset\subset \mathbb{Z}^d$

$$\boxed{\langle \sigma_A \rangle_S^{+} = \langle \sigma_A \mid \forall x \in \Lambda \setminus S \ \sigma_x = +1 \rangle_\Lambda^{+}.}$$

Pf: Let

$$F = \{ e \in E(\mathbb{Z}^d) : \ e \cap S = \emptyset \quad e \cap \Lambda \neq \emptyset \}$$



If $\sigma \in \Omega_\Lambda$ s.t. $\forall x \in \Lambda \setminus S \quad \sigma_x = +1$

$$H_\Lambda^{+}(\sigma) = -\beta \sum_{e \cap \Lambda \neq \emptyset} \sigma_e^{+}$$

$$= -\beta \sum_{e \cap S \neq \emptyset} \sigma_e^{+} - \beta \sum_{e \in F} \underbrace{\sigma_e^{+}}_{=1}$$

$$= H_S^{+}(\sigma|_S) - \beta |F|.$$

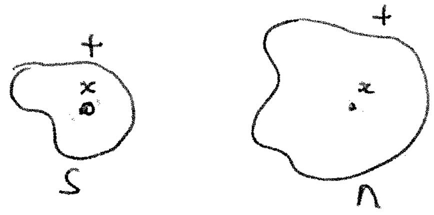
$$\langle \sigma_A \mid \forall x \in \Lambda \setminus S \ \sigma_x = +1 \rangle = \frac{\sum_{\sigma : \forall x \in \Lambda \setminus S \ \sigma_x = +1} \sigma_A e^{-H_S^{+}(\sigma|_S) + \beta |F|}}{\sum_{\sigma : \forall x \in \Lambda \setminus S \ \sigma_x = +1} e^{-H_S^{+}(\sigma|_S) + \beta |F|}}$$

$$= \langle \sigma_A \rangle_S^{+}. \quad \blacksquare$$

<u>Prop.</u> [Push. BC]

$$\boxed{\begin{array}{l}\text{Let } A \subset S \subset \Lambda \subset\subset \mathbb{Z}^d \quad \text{Then} \\[1em] \qquad \langle \sigma_A \rangle_S^+ \geq \langle \sigma_A \rangle_\Lambda^+ .\end{array}}$$

<u>Ex:</u> If $x \in S$



$$\langle \sigma_x \rangle_S^+ \geq \langle \sigma_x \rangle_\Lambda^+$$

"close BC have a bigger effect."

<u>Pf:</u> Observe that

$$f(\sigma) := \mathbb{1}_{\forall x \in \Lambda \setminus S \ \ \sigma_x = +1} = \prod_{x \in \Lambda \setminus S} \left(\frac{1+\sigma_x}{2} \right)$$

$$= \frac{1}{2^{|\Lambda \setminus S|}} \sum_{T \subset \Lambda \setminus S} \sigma_T$$

By the Lemma

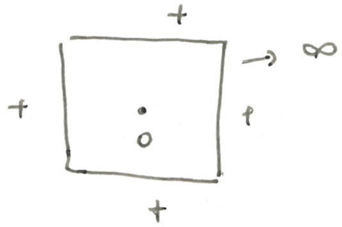
$$\langle \sigma_A \rangle_S^+ = \frac{\langle \sigma_A f \rangle_\Lambda^+}{\langle f \rangle_\Lambda^+} \overset{\text{GKS}}{\geq} \langle \sigma_A \rangle_\Lambda^+ \qquad \blacksquare$$

3 MAGNETIZATION AND CRITICAL PARAMETER

<u>Not</u>: $\Lambda_n = \{-n, \dots, n\}^d$

By the prop. in Sect. 2

$$\langle \sigma_0 \rangle^+_{\Lambda_{n+1}} \leq \langle \sigma_0 \rangle^+_{\Lambda_n}$$



<u>Def</u>: For $\beta \geq 0$, define   Well def. decreasing lim.

$$m(\beta) = m_d(\beta) := \lim_{n \to \infty} \langle \sigma_0 \rangle^+_{\Lambda_n, \beta}$$

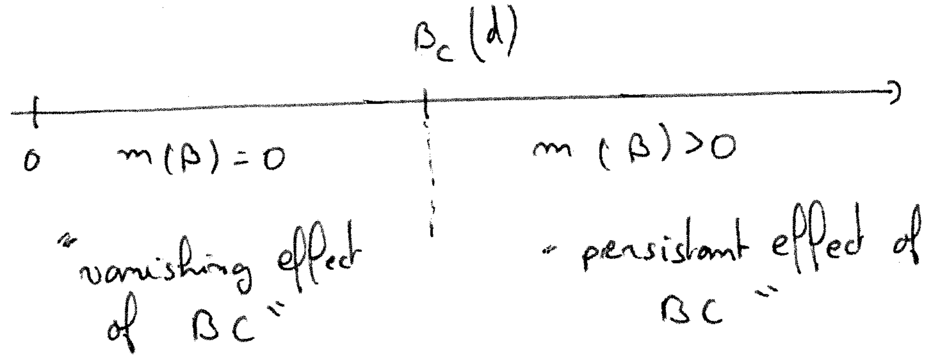
"magnetization at parameter β"

<u>Rk</u>: $m(\beta)$ is non decreasing in β.

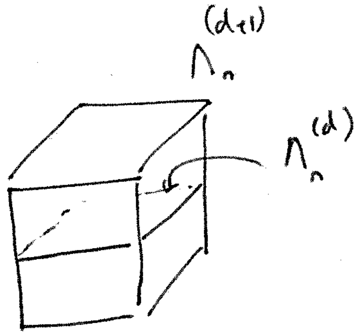
Indeed $$m(\beta) = \lim_{n \to \infty} \underbrace{\langle \sigma_0 \rangle^+_{\Lambda_n, \beta}}_{\nearrow \text{ in } \beta}.$$

Def. The critical parameter of the Ising model in dim d

$$\beta_c = \beta_c(d) = \sup\{\beta \geq 0 \text{ s.t. } m(\beta) = 0\}$$



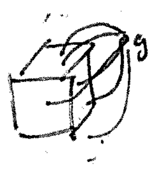
Prop. $\boxed{\beta_c(d+1) \leq \beta_c(d)}$



Pf: $\Lambda_n^{(d+1)} = \{-n, \dots, n\}^{d+1}$

$\Lambda_n^{(d)} = \{-n, \dots, n\}^{d} \times \{0\}$

(G, J) associated weighted graph to $\Lambda_n^{(d+1)}$



$$\tilde{J}_{xy} = \begin{cases} J_{xy} & \text{if } x, y \in \Lambda \text{ or } x \in \Lambda\ y = g \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \sigma_0 \rangle^+_{\Lambda_n^{(d)}} \underset{\substack{\uparrow \\ \text{exercise}}}{=} \langle \sigma_0 \rangle_{G, \tilde{J}} \leq \langle \sigma_0 \rangle_{G, J} = \langle \sigma_0 \rangle^+_{\Lambda_n^{(d+1)}}$$

Hence

$$\beta < \beta_c(d+1) \Rightarrow \lim_{n\to\infty} \langle \sigma_0 \rangle^{+}_{\Lambda_n^{(d+1)},\beta} = 0$$

$$\Rightarrow \lim_{n\to\infty} \langle \sigma_0 \rangle^{+}_{\Lambda_n^{(d)},\beta} = 0$$

$$\Rightarrow \beta \leq \beta_c(d)$$

$\square$

Rk: for $d=1$ we have seen

$$\forall \beta \geq 0 \quad \langle \sigma_0 \rangle^{+}_{\Lambda_n,\beta} \xrightarrow[n\to\infty]{} 0 \qquad m(\beta) = 0$$

$$\Leftrightarrow \boxed{\beta_c(d=1) = +\infty}$$

The goal in the rest of chapter will be to show that the phase transition is non-trivial if $d \geq 2$

Th: For every $d \geq 2$, we have

$$\boxed{0 < \beta_c(d) < \infty}$$

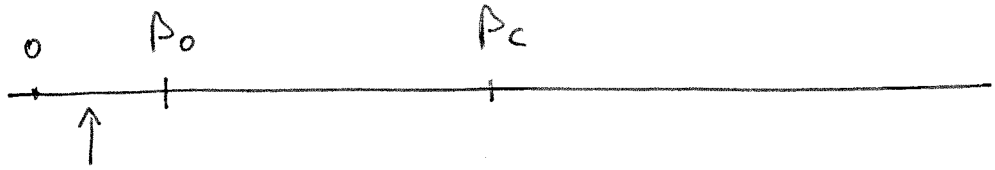
4 NO MAGNETIZATION AT HIGH TEMPERATURE

We prove that $\beta_c(d) > 0 \ \forall d \geq 1$

We actually prove the following stronger result:

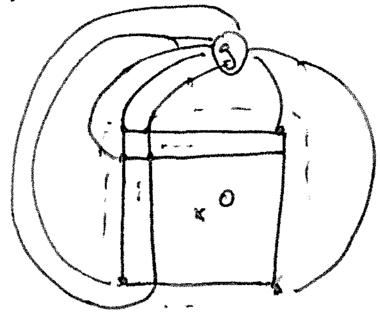
<u>Th</u>: $\forall \beta \leq \frac{1}{2de}$, we have

$$\boxed{\forall n \geq 1 \quad \langle \sigma_0 \rangle^+_{\Lambda_n, \beta} \leq e \times e^{-n}}$$



$\langle \sigma_0 \rangle^+_{\Lambda_n} \leq e^{-n}$

<u>Pf</u>: Consider the graph $G = (\Lambda_n \cup \{g\}, E)$ and J introduced in section 1

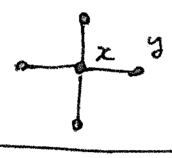


Set $\beta_0 =$

For $x \in \Lambda_{n-1}$, let $S = S_x = \{x\} \cup \partial\{x\}$

$\langle \sigma_x \sigma_y \rangle_S = \tanh(J_{xy}) = \tanh(\beta)$

$\leq \beta$



Hence $\boxed{\sum_{y \in \partial_{in} S} \langle \sigma_x \sigma_y \rangle \leq 2d\beta \leq \frac{1}{e}}$.

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By Simons inequality:

$\forall x \in \Lambda_{n-1}$

$$\langle \sigma_x \sigma_g \rangle \leq \sum_{y \in \partial_{in} S_x} \langle \sigma_x \sigma_y \rangle_{S_x} \langle \sigma_y \sigma_g \rangle$$

$$\leq \frac{1}{e} \max_{\substack{y \sim x \\ \text{"neigh"}}} \langle \sigma_y \sigma_g \rangle$$

$$\langle \sigma_0 \sigma_g \rangle \leq \frac{1}{e} \max_{x_1 \sim 0} \langle \sigma_{x_1} \sigma_g \rangle$$

$$\leq \frac{1}{e} \max_{x_1 \sim 0} \left(\frac{1}{e} \max_{x_2 \sim x_1} \langle \sigma_{x_2} \sigma_g \rangle \right)$$

$$\underset{\text{induct.}}{\leq} \frac{1}{e^k} \max_{x \in \Lambda_k} \langle \sigma_x \sigma_g \rangle$$

For $k=n$

$$\boxed{\langle \sigma_0 \sigma_g \rangle \leq e^{-n}}$$

$\hookrightarrow$ For. Ising on $\mathbb{Z}^d$, if $\beta \leq \frac{1}{2de}$

Then $\langle \sigma_0 \rangle^+_{\Lambda_n} \leq e^{-n}$ $\rightarrow$ no magnetization !