ISING MODEL

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Introduction

Initially introduced as a model for ferromagnetism, the *Ising Model* has become one of the most fundamental model in statistical mechanics with applications in several areas of science (thermodynamics, neuroscience, sociology,. . .).

1 Statistical mechanics

General idea Give a mathematical description of physical systems involving a *large number of elements*, such as:

- a glass of water ($\gg 10^{23}$ molecules)
- a piece of Iron ($\gg 10^{23}$ atoms)
- a population
- cars on a high way
- trees in a forest, . . .

Giving an exact description of such system is very hard. Let us take the example of a glass made of $N \gg 10^{23}$ molecules of water: one needs to keep track of the positions and speeds of all the molecules, which represent more than 10^{23} parameters! Instead, we give a probabilistic description, where each element is assumed to have a random behaviour, and the system is well described by *very few parameters*.

Modeling

$$\Omega = \{\text{"possible states of the systems"}\}$$
$$\mathrm{P}_{\beta_1,\dots,\beta_k} = \text{probability measure on } \Omega\text{, indexed by few parameters } \beta_1,\dots,\beta_k.$$

Typical questions We are interested in the large-scale behaviour of such system:

- Water: solid/liquid/gas
- Iron: paramagnetic/ferromagnetic

- population dyamic: survival/extinction?

For such systems, one often observes a sharp phase transition: a small change in the parameters gives rise to different macroscopic behaviours (think of water at 0°C).

2 Para/ferromagnetic phase transition

Ising model was introduced by Wilhelm Lenz in 1920 in view of a theoretical understanding of the para/ferromagnetic phase transition. The model was named after Ernst Ising (Lenz's student) who studied the one dimensional version of the model in his PhD thesis (1925). In this section we give a brief description of the para/ferromagnetic phase transition.

Microscopic description Each atom of iron has a magnetic moment, and can be seen as a small magnet. For simplicity, the moment is assumed to point in two possible and opposite directions (spin up and spin down). These small magnets interact with each other, and two neighbouring atoms prefer to have aligned spins. The strength of these interactions is inverse proportional to the temperature.

Curie temperature The piece of Iron undergoes a phase transition at a critical temperature

$$T_c = T_c(\mathrm{Fe}) = 1034K,$$

called the Curie temperature (Pierre Curie 1859-1906).

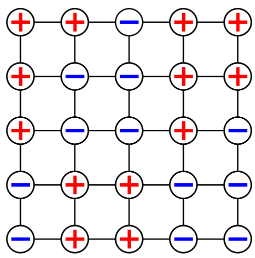
At low temperature ($T < T_c$), the interaction is strong and a majority of spin point towards the same direction (either up or down)

At high temperature ($T > T_c$) the interaction is weak and no spin is preferred globally. Spins up and down coexist in equal proportion.

3 Modelization: Boltzman formalism

We present the formal definition of Ising model in a box of dimension $d \geq 1$.

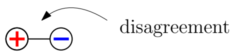
$$\Lambda = \Lambda_n = \{-n, \dots, n\}^d$$



Spin configuration: $\sigma = (\sigma_x)_x \in \Lambda \in \{-1,+1\}^\Lambda$

$$\begin{aligned} \sigma_x &= +1 \quad \text{"spin up"} \\ \sigma_x &= +1 \quad \text{"spin down"} \end{aligned}$$

Goal Define a probability measure μ_β on the state space $\{-1,+1\}^\Lambda$ which favors configurations with few disagreements between neighbours (A particle tries to have the same spin as its neighbours).



The underlying parameter $\beta \geq 0$ correspond to the strength of the interaction.

Energy of a configuration For $\beta \geq 0$ and $\sigma \in \{-1,+1\}^\Lambda$, let

$$\boxed{H_\beta(\sigma) = -\beta \sum_{x,y \text{ neighbours}} \sigma_x \sigma_y}$$

Remark 3.1. *We have $H_\beta(\sigma) = \beta \sum_{x,y \text{ neigh.}} (2\mathbf{1}_{\sigma_x \neq \sigma_y} - 1)$. The energy of σ is low when the number of disagreement is low. In particular the lowest energy correspond to the two constant states*

$$\sigma = \overline{1} \quad \textit{and} \quad \sigma = \overline{-1}$$

which are the only configurations with no disagreements. These two minimizers are called the ground states *associated to H_β.*

Probability of a configuration For $\beta \geq 0$ and $\sigma \in \{-1,+1\}^\Lambda$, let

$$\boxed{\mu_\beta(\sigma) = \frac{1}{Z_\beta} e^{-H_\beta(\sigma)}} \quad \text{where } Z_\beta = \sum_{\sigma \in \{\pm 1\}^\Lambda} e^{-H_\beta(\sigma)}$$

The constant Z_β is called the *partition function.* It is defined in such a way that μ_β is a probability measure.

Remark 3.2. *If σ has a high energy $H_\beta(\sigma)$, then its probability $\mu_\beta(\sigma)$ is small.*

Effect of the parameter β

➜ For $\beta = 0$, we have

$$\forall \sigma \in \{-1,+1\}^{|\Lambda|} \quad \mu_{\beta=0}(\sigma) = \frac{1}{2^{|\Lambda|}}.$$

When $\beta = 0$, all the configurations have the same probability. The spins behave independently and there is no interaction.

➜ When $\beta \to \infty$, we have

$$\lim_{\beta \to \infty} \mu_\beta(\sigma) = \begin{cases} \frac{1}{2} & \text{if } \sigma = \overline{1} \text{ or } \sigma = \overline{-1} \\ 0 & \text{otherwise.} \end{cases}$$

The Ising measure at $\beta = \infty$ concentrates on the two ground states: all the spins are aligned.

As β varies continuously from 0 to infinity, the interaction between the spins increase, which favors the alignment between them.

In physics, the Ising model is rather parametrized by the temperature T, which is related to β by the formula

$$\beta = \frac{k}{T},$$

where k is the Boltzman constant. At low temperature, the spins are more aligned than at high temperature, where the thermal agitation reduces the interactions between the atoms.

4 Magnetization and phase transition

Write $\langle \cdot \rangle_{\Lambda_n,\beta}$ for the expectation corresponding to the measure μ_β introduced in the previous section. Under the Ising measure μ_β, the expected spin at 0 is

$$\langle \sigma_0 \rangle_{\Lambda_n,\beta} = 0$$

because $\mu_\beta(\sigma) = \mu_\beta(-\sigma)$ (spin-flip symmetry). In order to "break" this symmetry, we consider the measure with + boundary conditions (which corresponds to conditioning all spins equal to +1 at the boundary of the box).

$$\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} = \langle \sigma_0 \mid \forall x \in \Lambda_{n+1} \backslash \Lambda_n \ \sigma_x = +1 \rangle_{\Lambda_{n+1},\beta}.$$

In the first part of the course we will prove the following properties.

Positive effect of the + boundary conditions The + boundary conditions may only increase the chance for the spin σ_0 to be positive: for every $n \geq 0$ and every $\beta \geq 0$, we have

$$\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \geq 0. \tag{P1}$$

Pushing boundary conditions The effect of the boundary conditions is weaker when they are pushed away: for every $n \geq 0$ and every $\beta \geq 0$, we have

$$\langle \sigma_0 \rangle^+_{\Lambda_{n+1},\beta} \leq \langle \sigma_0 \rangle^+_{\Lambda_n,\beta}. \tag{P2}$$

Monotonicity in the parameter β If the strength of the interaction β increases, then the spins get more aligned, and the spin at 0 has more chance to be aligned to +1: for every $n \geq 0$ and every $\beta' \geq \beta \geq 0$, we have

$$\langle \sigma_0 \rangle^+_{\Lambda_n,\beta'} \geq \langle \sigma_0 \rangle^+_{\Lambda_n,\beta}. \tag{\textbf{P3}}$$

The property (**P2**) allows us to define the magnetization as a monotone limit as follows.

Definition 4.1 (Magnetization). *For every $\beta \geq 0$, define*

$$m(\beta) := \lim_{n\to\infty} \langle \sigma_0 \rangle^+_{\Lambda_n,\beta}.$$

Properties (**P1**) and (**P3**) imply that the magnetization $m(\beta)$ is nonnegative and nondecreasing in the parameter β. A central question is whether we have $m(\beta) > 0$ (the effect of the $+$ boundary conditions pertains when they are pushed away to infinity) or $m(\beta) = 0$ (the effect of the boundary conditions vanishes at infinity). The answer to this question depends on the unerlying paprameter β, and lead to the definition of the phase transition for Ising model.

Definition 4.2 (Critical parameter for Ising model).

$$\beta_c := \sup\{\beta \geq 0 \ s.t. \ m(\beta) = 0\}$$

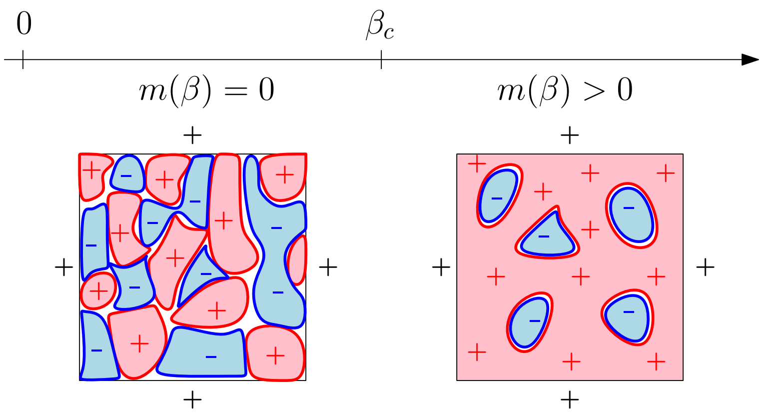


Figure 0.1: Illustration of the phase transition. For $\beta < \beta_c$, the interaction is weak, the $+$ and - phases are mixed together. For $\beta > \beta_c$, the interaction is strong, a mojority of spins align to +1.

In the course we will prove that

- $\beta_c = +\infty$ in dimension $d = 1$ (no phase transition).
- $0 < \beta_c < \infty$ in dimension $d \geq 2$ (non trivial phase transition)

Chapter 1

Ising measure on a finite set

Goals:

- ➜ Define the Ising model in the abstract setting of finite weighted graphs with general weights.
- ➜ Define and discuss the n-point function.
- ➜ Define the ghost versions of the model.
- ➜ Relate the abstract framework to the more standard version of the model (lattice case, external field, Curie-Weiss model).

1 Graph theoretical framework

Definition 1.1. *A* finite graph *with vertex set V and edge set E is a pair $G = (V, E)$, where V is a finite set, and $E \subset \{\{x,y\} : x,y \in V, x \neq y\}$.*

We emphasize that in our convention, a finite graph has no self-loop, and no multiple edges.

Notation: For $x,y \in V$, we write $xy := \{x,y\}$.

Let $G = (V,E)$ be a finite graph. Let $x,y \in V$. Let $\ell \in \mathbb{N}$. A *path of length ℓ from x to y* is a sequence of distinct vertices $\gamma = (\gamma_0, \ldots, \gamma_\ell)$ such that $\gamma_i\gamma_{i+1} \in E$ for every $1 \leq i < \ell$. Given such a path, we write $e \in \gamma$ if there exists i such that $e = \gamma_i\gamma_{i+1}$. When such a path exists, we say that x and y are connected in G.

Definition 1.2. *Let $G = (V,E)$ be a finite graph. We call* weights on G *a collection of real numbers $J = (J_e)_{e \in E}$ satisfying $J_e \geq 0$ for every edge e. Given such weights, we write $G[J > 0]$ for the subgraph of G, with vertex set V, and edge set $\{e \in E : J_e > 0\}$.*

Remark 1.3. *In the physics literature, weights satisfying J_e are said to be* ferromagnetic *since they correspond to a positive interaction of the spins.*

From now on and until the end of the chapter, we fix

- a finite graph $G = (V, E)$,
- some weights $J = (J_e)_{e \in E}$.

2 Probabilistic framework

Ising measure We call *spin configuration* an element σ of the product set

$$\Omega := \{+1, -1\}^V.$$

To each spin configuration $\sigma \in \Omega$ we associate an *energy* defined by

$$\boxed{H(\sigma) = -\sum_{xy \in E} J_{xy}\sigma_x\sigma_y}.$$

Finally, the *Ising measure on* (G, J) is the probability measure μ on Ω defined by

$$\forall \sigma \in \Omega \quad \boxed{\mu(\sigma) = \frac{1}{Z} e^{-H(\sigma)},}$$

where the normalizing constant $Z = \sum_{\sigma \in \Omega} e^{-H(\sigma)}$ is called the *partition function*.

Random variables and expectation We have now a probability space (Ω, μ), which gives rise to the standard notion of probability, such as random variables, expectation,...

The expectation of a real random variable $f : \Omega \to \mathbb{R}$ is denoted by

$$\langle f \rangle = \frac{1}{Z} \sum_{\sigma \in \Omega} f(\sigma) e^{-H(\sigma)}.$$

Occasionally it may also be convenient to consider the integral of f without normalization, and we intoduce

$$Z[f] := \sum_{\sigma \in \Omega} f(\sigma) e^{-H(\sigma)}.$$

Spin-flip symmetry The measure μ satisfies the symmetry $\mu(\sigma) = \mu(-\sigma)$. In other words, σ and $-\sigma$ have the same law under μ.

If f is an odd random variable (i.e. $f(\sigma) = -f(-\sigma)$ for every configuration σ), then

$$\langle f \rangle = \sum_{\sigma \in \Omega} f(\sigma)\mu(\sigma) = \sum_{\sigma \in \Omega} f(-\sigma)\mu(-\sigma) = \sum_{\sigma \in \Omega} -f(\sigma)\mu(\sigma) = -\langle f \rangle,$$

which implies that $\langle f \rangle = 0$.

3 Examples

Several examples fit in the general framework of Section 2.

Curie-Weiss model Let $N \geq 1$. The Curie-Weiss model corresponds to an Ising model on the complete graphs with N vertices. The coupling constant scaled in such a way the limit of the model is non trivial as N tends to infinity. It is defined by choosing

$$\begin{aligned} V &= \{1, \dots, N\} \\ E &= \{xy : x, y \in V\} \\ J_e &= \frac{\beta}{N} \quad \text{for every } e \in E. \end{aligned}$$

See [FV18, Chapter 2] for a detailed study of Curie-Weiss model.

Box in $\mathbb{Z}^d$ The lattice version of Ising model (in $\mathbb{Z}^d$, $d \geq 1$) discussed in the introduction also fits in the framework. Given a box size $n \geq 1$ and an inverse temperature $\beta \geq 0$, it corresponds to the weighted graph given by

$$\begin{aligned} V &= \{-n, \dots, n\}^d, \\ E &= \{xy : \|x - y\|_1 = 1\} \\ J_e &= \beta \quad \text{for every } e \in E. \end{aligned}$$

4 Multi-point functions

Definition 4.1. *For every set $A \subset V$ we set*

$$\sigma_A = \prod_{x \in A} \sigma_x$$

(identified with the random variable $\sigma \mapsto \prod_{x \in A} \sigma_x$).

For every $k \geq 2$, the mapping $A \mapsto \langle \sigma_A \rangle$, restricted to sets A of size k, is sometimes called the *k-point function.*

Remark 4.2 (Odd sets)**.** *The spin-flip symmetry implies that*

$$\boxed{\forall A \subset V \ \text{odd} \quad \langle \sigma_A \rangle = 0.} \tag{1.1}$$

Lemma 4.3. *For every $A, B \subset V$, we have*

$$\frac{1}{|\Omega|} \sum_{\sigma \in \Omega} \sigma_A \sigma_B = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{if } A \neq B. \end{cases}$$

Probabilistic proof. Let $\mathbf{E}$ be the expectation on Ω w.r.t. the uniform measure. Notice that under the uniform measure all the individual spins σ_x are independent. Hence, for every $A, B \subset V$, we have

$$\mathbf{E}[\sigma_A \sigma_B] = \mathbf{E}[\sigma_{A\Delta B}] = \prod_{x \in A\Delta B} \mathbf{E}[\sigma_x] = \begin{cases} 0 & \text{if } A \neq B, \\ 1 & \text{if } A = B. \end{cases}$$

□

Combinatorics proof. Assume $A \neq B$, and fix $x \in A\Delta B$. For every σ define

$$\widetilde{\sigma}_y = \begin{cases} \sigma_y & \text{if } y \neq x \\ -\sigma_y & \text{if } y = x. \end{cases}$$

Notice that $\sigma \mapsto \widetilde{\sigma}$ is a bijection of Ω to itself, and that $\widetilde{\sigma}_A \widetilde{\sigma}_B = -\sigma_A \sigma_B$. Hence

$$\sum_{\sigma \in \Omega} \sigma_A \sigma_B = \sum_{\sigma \in \Omega} \widetilde{\sigma}_A \widetilde{\sigma}_B = -\sum_{\sigma \in \Omega} \sigma_A \sigma_B.$$

□

Proposition 4.4. *$(\sigma_A)_{A \subset V}$ forms a basis of the space of real random variables $\mathbb{R}^\Omega$.*

Proof. First, observe that the family $(\sigma_A)_{A\subset V}$ as $2^{|V|} = \dim(\mathbb{R}^\Omega)$ elements. Second, by Lemma 4.3, it is orthonormal family with respect to the inner product defined

$$(f;g) = \frac{1}{|\Omega|} \sum_{\sigma \in \Omega} f(\sigma) g(\sigma).$$

Therefore, the functions σ_A, $A \subset V$ are linearly independent, which completes the proof. □

Remark 4.5. *The family (σ_A) appears naturally as the orthonormal basis of characters when one studies Fourier analysis on the finite group $\{+1,-1\}^V$, see e.g.[GS15].*

A direct consequence of the proposition above is that any random variable f can be decomposed as a linear sum $f = \sum_{A \subset V} f_A \sigma_A$, $f_A \in \mathbb{R}$. Therefore, the measure μ is characterized by $(\langle \sigma_A \rangle)_{A \subset V}$.

5 Ising model with a ghost

Fix a vertex $g \in V$, called the *ghost* vertex. We define the + Ising measure on (V, J, g) by

$$\mu^+ := \mu[\,\cdot\,|\sigma_g = +1].$$

This measure allows us to "break" the spin-flip symmetry, and appears more often than the unconditioned measure μ in applications: it will be particularly useful when we introduce + boundary conditions or external fields, as in the examples below.

Example 1: Box in $\mathbb{Z}^d$ with + boundary conditions Let $d \geq 1$, $n \geq 1$, $\beta \geq 0$. Write $\Lambda = \{-n, \dots, n\}^d$, and consider an abstract symbol $g \notin \Lambda$.

The Ising model in a finite box with + boundary condition corresponds to the measure μ^+ with the choice:

$$
\begin{aligned}
&V = \Lambda \cup \{g\},\\
&E = \{xy \,:\, x, y \in \Lambda,\ \|x - y\|_1 = 1\} \cup \{xg \,:\, x \in \Lambda,\ \|x\|_\infty = n\},\\
&J_e = \beta \qquad \text{for every } e \in E.
\end{aligned}
$$

Example 2: Box in $\mathbb{Z}^d$ with external field As above, we consider $d \geq 1$, $n \geq 1$, $\beta \geq 0$, and $g \notin \Lambda$. We also consider a nonnegative $h \geq 0$. The Ising model in a finite box with external magnetic field h corresponds to the measure μ^+ with the choice:

$$
\begin{aligned}
&V = \Lambda \cup \{g\},\\
&E = \{xy \,:\, x, y \in \Lambda,\ \|x - y\|_1 = 1\} \cup \{xg \,:\, x \in \Lambda\},\\
&J_e = \begin{cases} \beta & \text{if } g \notin e,\\ h & \text{if } g \in e. \end{cases}
\end{aligned}
$$

Relation to the unconditioned measure. In general, conditioned measure are more delicate to study than the original one. Here, μ^+ is not really more complicated than μ, and inherits several properties form μ, thanks to the spin-flip symmetry. For example, all the multi-point functions for μ^+ can be easily computed from those of μ.

Proposition 5.1 (multi-point functions for + Ising model)**.** *Write $\langle\cdot\rangle^+$ for the expectation associated to the measure μ^+. For every set $A \subset V$, we have*

$$
\boxed{\langle \sigma_A \rangle^+ = \begin{cases} \langle \sigma_{A \Delta \{g\}} \rangle & \text{if } A \text{ is odd,}\\ \langle \sigma_A \rangle & \text{if } A \text{ is even.} \end{cases}}
$$

Proof. By spin-flip symmetry we have $\mu[\sigma_g = 1] = \frac{1}{2}$. Therefore,

$$
\langle \sigma_A \rangle^+ = 2\langle \sigma_A \ \mathbf{1}_{\sigma_g = 1} \rangle = \langle \sigma_A \ (\sigma_g + 1) \rangle = \langle \sigma_{A \Delta \{g\}} \rangle + \langle \sigma_A \rangle.
$$

We conclude the proof by using Eq. (1.1), which implies that either first or the second term vanishes in the last sum, depending whether the set A is even or odd, respectively. □

Chapter 2

Random current representation

Let $x, y \in V$. It is natural to expect that the two spins σ_x and σ_y are positively correlated, i.e. $\langle\sigma_x\sigma_y\rangle \geq 0$. This does not follow from the definition of the expectation

$$\langle\sigma_x\sigma_y\rangle = \sum_{\sigma\in\Omega} \sigma_x\sigma_y e^{-H(\sigma)},$$

since the sum has positive and negative terms. In this chapter, we introduce the random current representation, a powerful tool to study the multi-spin function, based on a percolation-type interpretation of the model.

Goals:

➜ Introduce the concept of geometric representation

➜ Define the random current representation (RC)

➜ Express $\langle\sigma_A\rangle$ via the RC.

Framework

- (V, J) finite weighted graph with ferromagnetic coupling constants. $E = \{xy : x, y \in V\}$.
- $\Omega = \{-1, +1\}^V$ space of spin configurations, equipped with the (free) Ising measure μ:

$$H(\sigma) = -\sum_{xy\in E} J_{xy}\sigma_x\sigma_y \quad \longrightarrow \quad \mu[\sigma] = \frac{1}{Z}e^{-H(\sigma)}.$$

1 Currents and their sources

Definition 1.1. *A* current on G *is a function*

$$\mathbf{n} : E \to \mathbb{N}.$$

For every $x \in V$, we write $\deg_{\mathbf{n}}(x) := \sum_{e \ni x} \mathbf{n}_e$ *and define the* sources of $\mathbf{n}$ *as the subset of vertices*

$$\boxed{\partial \mathbf{n} := \{x \in V \,:\, \deg_{\mathbf{n}}(x) \text{ odd}\}.}$$

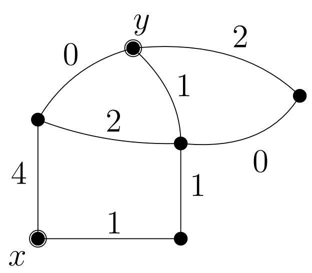


Figure 2.1: A current with $\partial \mathbf{n} = \{x, y\}$.

Remark 1.2. *We have*

$$\sum_{x \in V} \deg_{\mathbf{n}}(x) = 2 \sum_{e} \mathbf{n}_e,$$

therefore the set of sources $\partial \mathbf{n}$ is always even.

We are particularly interested in the connectivity properties of a current. For $x, y \in V$, we write

$$x \xleftrightarrow{\mathbf{n}} y$$

if x and y are connected in the graph $G[\mathbf{n} > 0]$. In this case we say that x and y are *connected in* $\mathbf{n}$. The *cluster of x in* $\mathbf{n}$ is the subset of vertices defined by

$$C_x(\mathbf{n}) := \{y \in V \,:\, x \xleftrightarrow{\mathbf{n}} y\}.$$

The following combinatorial property of currents will be used repeatedly.

Proposition 1.3 (connectivity of sources)**.** *Let $\mathbf{n}$ be a current on E . For every source $x \in \partial \mathbf{n}$, there exists another source $y \in \partial \mathbf{n} \setminus \{x\}$ such that $x \xleftrightarrow{\mathbf{n}} y$.*

Proof. Let $x \in \partial \mathbf{n}$. Notice that all the edges e at the boundary of $C_x(\mathbf{n})$ (i.e. the edges with exactly one extremity on $C_x(\mathbf{n})$) satisfy $\mathbf{n}_e = 0$. Hence

$$\sum_{y \in C_x(\mathbf{n})} \deg_{\mathbf{n}}(y) = 2 \sum_{e \subset C_x(\mathbf{n})} \mathbf{n}_e.$$

Since $\deg_{\mathbf{n}}(x)$ is odd, there must exist $y \in C_x(\mathbf{n}) \setminus \{x\}$ such that $\deg_{\mathbf{n}}(y)$ is odd. $\square$

This has the following consequences:

Corollary 1.4. *Let* $\mathbf{n} \in \mathbb{N}^E$ *be a current.*

- *If* $\mathbf{n}$ *has exactly two sources* $\partial\mathbf{n} = \{x, y\}$ *then we have* $x \overset{\mathbf{n}}{\longleftrightarrow} y$.
- *In general, if* $\mathbf{n}$ *has* $2k$ *sources,* $k \in \mathbb{N}$*one can always pair them* $\mathbf{n} = \{x_i\}_{1\leq i\leq k} \cup \{y_i\}_{1\leq i\leq k}$ *in such a way that* $x_i \overset{\mathbf{n}}{\longleftrightarrow} y_i$ *for every* i.

2 Poisson point process on the edges

Definition 2.1. *We call Poisson point process on* E *with intensity* J *(ppp(*J*)) a sequence of independent random variables* $(N_e)_{e\in E}$*, where*

$$N_e \sim \text{Poisson}(J_e).$$

The terminology "Poisson point process" comes from the interpretation of J as a measure on E, where J_e corresponds to the measure of the singleton $\{e\}$. And equivalently, N can be seen as a random measure on E, which is exactly a Poisson point process on E, with intensity measure J.

Proposition 2.2 (Thinning)**.** *Let* K *be a ppp(*$2J$*). Independently of* K*, Let* $Z = (Z_e^i)_{e\in E, i\in\mathbb{N}}$ *be a collection of iid Bernoulli(*$1/2$*) random variable. Then* M *and* N *defined by*

$$\forall e \in E \quad M_e = \sum_{i=1}^{K_e} Z_e^i \quad \text{and} \quad N_e = K_e - M_e$$

are two independent ppp(J*).*

3 Random Current representation of the multi-spin function

Lemma 3.1 (Expansion of products)**.** *Let* $f : E \times \mathbb{N} \to \mathbb{R}$ *and assume that*

$$\forall e \in E \quad \sum_{n\in\mathbb{N}} |f(e,n)| < \infty.$$

Then, we have

$$\prod_{e\in E} \Big(\sum_{n\in\mathbb{N}} f(e,n) \Big) = \sum_{\mathbf{n}\in\mathbb{N}^E} \Big(\prod_{e\in E} f(e, \mathbf{n}_e) \Big)$$

and the sum on the right hand side is absolutely convergent.

Proof. We may assume without loss of generality that $E = \{1, \dots, k\}$, and then we prove the result by induction on k. For $E = \{1\}$, there is nothing to do. Let $k \geq 1$ and assume that the result holds for $E = \{1, \dots, k\}$. Now, consider

$$f : \{1, \dots, k+1\} \times \mathbb{N} \to \mathbb{R}$$

such that the $k+1$ series $\sum_n f(i,n)$, $i = 1, \dots, k+1$ are absolutely convergent. By first applying the induction hypothesis, and then using Fubini's theorem to combine the double sum into one, we find

$$\begin{aligned}
\prod_{i \leq k+1} \Big(\sum_{n \in \mathbb{N}} f(i,n) \Big) &= \Bigg(\sum_{(n_1, \dots, n_k) \in \mathbb{N}^k} f(1,n_1) \cdots f(k,n_k) \Bigg) \Bigg(\sum_{n \in \mathbb{N}} f(k+1,n) \Bigg) \\
&= \sum_{(n_1, \dots, n_k) \in \mathbb{N}^k} \Bigg(\sum_{n_{k+1} \in \mathbb{N}} f(1,n_1) \cdots f(k+1,n_{k+1}) \Bigg) \\
&= \sum_{(n_1, \dots, n_{k+1}) \in \mathbb{N}^{k+1}} f(1,n_1) \cdots f(k+1,n_{k+1}),
\end{aligned}$$

and the last sum is absolutely convergent by Fubini's theorem. □

Theorem 3.2 (Random-current representation of the multipoint functions)**.** *Let N be a ppp(J) on E. For every $A \subset V$, we have*

$$\boxed{\langle \sigma_A \rangle = \frac{\mathbb{P}[\partial N = A]}{\mathbb{P}[\partial N = \varnothing]}} \tag{2.1}$$

Proof. We first fix $\sigma \in \Omega$, and expand the weight $e^{-H(\sigma)}$ as a sum. To achieve this, we first write this weight as a product of exponential terms over the edges, then we write the exponential terms as series, and finally expand the product via Lemma 3.1. This gives

$$\begin{aligned}
e^{-H(\sigma)} = \exp\big(\sum_{e \in E} J_e \sigma_e\big) = \prod_{e \in E} \exp(J_e \sigma_e) &= \prod_{e \in E} \Big(\sum_{n \in \mathbb{N}} \frac{1}{n!} (J_e \sigma_e)^n \Big) \\
&= \sum_{\mathbf{n} \in \mathbb{N}^E} \Big(\prod_{e \in E} \frac{1}{\mathbf{n}_e!} (J_e \sigma_e)^{\mathbf{n}_e} \Big) = \sum_{\mathbf{n} \in \mathbb{N}^E} \Big(\prod_{e \in E} \frac{J_e^{\mathbf{n}_e}}{\mathbf{n}_e!} \Big) \Big(\prod_{e \in E} \sigma_e^{\mathbf{n}_e} \Big).
\end{aligned}$$

In the last equation the first product is equal to $e^{|J|}$ times the probability that a ppp(J) is equal to $\mathbf{n}$ (where $|J| := \sum_{e \in E} J_e$). In the second product each factor σ_x, $x \in V$ appears raised to the power $\sum_{e \ni x} \mathbf{n}_e$. Hence σ_x does not disappear only if x is a source of $\mathbf{n}$, and we have

$$\prod_{e \in E} \sigma_e^{\mathbf{n}_e} = \sigma_{\partial \mathbf{n}}.$$

Plugging this expression in the equation above, we obtain for every $\sigma \in \Omega$

$$e^{-H(\sigma)} = e^{|J|} \sum_{\mathbf{n} \in \mathbb{N}^E} \mathbb{P}[N = \mathbf{n}] \cdot \sigma_{\partial \mathbf{n}}.$$

By using this expression of the weight, and permuting the sums, we obtain for every $A \subset V$

$$Z[\sigma_A] = \sum_{\sigma \in \Omega} \sigma_A \cdot e^{-H(\sigma)} = e^{|J|} \sum_{\mathbf{n} \in \mathbb{N}^E} \mathbb{P}[N = \mathbf{n}] \cdot \sum_{\sigma \in \Omega} \sigma_A \sigma_{\partial \mathbf{n}}.$$

Therefore, by Lemma 4.3, we find

$$Z[\sigma_A] = e^{|J|} \cdot |\Omega| \cdot \mathbb{P}[\partial N = A].$$

Finally we conclude the proof by using that $\langle \sigma_A \rangle = \frac{Z[\sigma_A]}{Z[1]}$. □

Remark 3.3. *The denominator is always positive since*

$$\mathbb{P}[\partial N = \varnothing] \geq \mathbb{P}[\forall e \in E \;\; N_e = 0] = e^{-|J|} > 0.$$

In contrast, the numerator may vanish, for example when A is odd.

A direct consequence of random current representation is that the multipoint spin function is always nonnegative, since it is the ratio of two probabilities..

Corollary 3.4 (First GKS inequality)**.** *For every $A \subset V$, we have*

$$\langle \sigma_A \rangle \geq 0.$$

This inequality is fundamental in the theory and has many consequences, some of them will be discuss in chapter 4.

Exercise 3.5. *Let $A \subset V$. When do we have $\langle \sigma_A \rangle > 0$?*

Remark 3.6. *The left hand side in* (2.1) *is at most* 1*, and therefore the theorem implies that $\mathbb{P}[\partial N = A] \leq \mathbb{P}[\partial N = \varnothing]$.*

4 Application: exact computations in dimension 1

In the introduction, we defined the magnetization in $\mathbb{Z}$ as

$$m(\beta) = \lim_{n \to \infty} \langle \sigma_0 \mid \sigma_{-n} = \sigma_n = +1 \rangle^+_{\Lambda_n}, \tag{2.2}$$

where $\mu^+_{\Lambda_n}$ is the Ising measure on $\Lambda_n = \{-n, \ldots, n\}$ with $E = \{xy \,:\, |x - y| = 1\}$ weights $J_e = \beta$.

Consider the graph G_n obtained from Λ_n by identifying the two vertices $-n$ and $+n$ into a new vertex called g. The graph G_n corresponds to the cyclic weighted graph illustrated below. As discussed in the previous chapter, on can reinterpret the term in the right hand side of (2.2) as the expectation of σ_0 wrt to the +-Ising measure on G_n, with a (single) ghost at g.

$$\langle \sigma_0 \mid \sigma_{-n} = \sigma_n = +1 \rangle_{\Lambda_n} = \langle \sigma_0 \rangle^+_{G_n}.$$

By Proposition 5.1, we have

$$\langle \sigma_0 \rangle^+_{G_n} = \langle \sigma_0 \sigma_g \rangle_{G_n},$$

where μ_{G_n} is the (free) Ising measure on G_n. By using the random current representation, we have

$$\langle \sigma_0 \sigma_g \rangle_{G_n} = \frac{\mathbb{P}[\partial N = 0g]}{\mathbb{P}[\partial N = \varnothing]}.$$

Notice that a Poisson random variable is even with probability

$$\sum_{n \in 2\mathbb{N}} e^{-\beta} \frac{\beta^n}{n!} = e^{-\beta} \cosh \beta,$$

and odd with probability $e^{-\beta} \sinh \beta$. The random current N is sourceless if all the weights N_e are of the same parity. By independence, we obtain

$$\mathbb{P}[\partial N = \varnothing] = e^{-2n\beta} (\sinh^{2n} \beta + \cosh^{2n} \beta).$$

Similarly, by examining the parity constraint imposed by sources at 0 and g, we find

$$\mathbb{P}[\partial N = 0g] = 2e^{-2n\beta} (\sinh^{n} \beta \cosh^{n} \beta).$$

Finally, we obtain

$$\lim_{n \to \infty} \langle \sigma_0 \mid \sigma_{-n} = \sigma_n = +1 \rangle^+_{\Lambda_n} = \frac{\sinh^n \beta \cosh^n \beta}{\sinh^{2n} \beta + \cosh^{2n} \beta},$$

which is asymptotically equivalent to $\sinh^n \beta$ as n tends to infinity. This concludes that $m(\beta) = 0$ for every $\beta \geq 0$, and therefore $\beta_c = \infty$ in dimension 1.

Chapter 3

Double random current

In the previous chapter, we express $\langle\sigma_A\rangle$ as the ratio

$$\frac{\mathbb{P}[\partial N = A]}{\mathbb{P}[\partial N = \varnothing]}$$

The two probabilities appearing in the ratio are "degenerated" in the sense that they are exponentially small in $|J|$ (in general). Therefore, they do not give a "nice" probabilistic interpretation of $\langle\sigma_A\rangle$. In this chapter, we will see that one can obtain a better probabilistic interpretation of $\langle\sigma_A\rangle^2$. We will use a duplication principle (use two independent currents rather than one) and then a manipulation on currents, called the switching lemma, that will allow us to switch the sources from one current to the other..

Goals:

- ➜ Introduce the duplication method in order to express products of the form $\langle\sigma_A\rangle\langle\sigma_B\rangle$.
- ➜ Switching Lemma for sources in a double random current.
- ➜ Give a probabilistic interpretation of $\langle\sigma_A\rangle^2$ in term of a double current.
- ➜ Prove the second GKS inequality, monotonicity in J, and Simon-Lieb inequality.

Framework

- $G = (V, E)$ finite graph with $(J_e)_{e\in E}$ nonnegative weights.
- $\Omega = \{-1, +1\}^V$ space of spin configurations, equipped with the Ising measure μ:

$$H(\sigma) = -\sum_{xy\in E} J_{xy}\sigma_x\sigma_y \quad\longrightarrow\quad \mu[\sigma] = \frac{1}{Z}e^{-H(\sigma)}.$$

1 Switching lemma

Consider the set of current

$$\mathcal{F}_A = \{\mathbf{n} \in \mathbb{N}^E : |C \cap A| \text{ is even for every cluster } C \text{ of } \mathbf{n}\}$$

Notice that the event $\mathcal{F}_A$ is empty if A is odd.

Lemma 1.1. *Let $A \subset V$. Let $\mathbf{m}, \mathbf{n} \in \mathbb{N}^E$ be two currents*

1. *If $\partial\mathbf{n} = A$ then $\mathbf{n} \in \mathcal{F}_A$.*
2. *If $\mathbf{m} \in \mathcal{F}_A$ and $\mathbf{m} \le \mathbf{n}$ then $\mathbf{n} \in \mathcal{F}_A$.*
3. *If $\mathbf{n} \in \mathcal{F}_A$, there exists $\eta = \eta(\mathbf{n}) \in \{0,1\}^E$ such that $\eta \le \mathbf{n}$ and $\partial\eta = A$.*

Proof. The first item follows from Corollary 1.4. For the second item, consider a cluster C of $\mathbf{n}$. Since $\mathbf{m} \le \mathbf{n}$, this cluster can be decomposed as a disjoint union

$$C = C_1 \cup \cdots \cup C_k,$$

where $k \ge 1$ and $C_1, \dots, C_k$ are disjoint clusters of $\mathbf{m}$. Therefore $|C \cap A| = |C_1 \cap A| + \cdots + |C_k \cap A|$ is even. We now move the third item. Without loss of generality we may assume that $|A| = 2k$ for some integer $k \ge 0$. Let $\mathbf{n} \in A$. Since each cluster of $\mathbf{n}$ intersects A at an even number of points, one can pair the elements of A as

$$A = \{x_1, \dots, x_k\} \cup \{y_1, \dots, y_k\}$$

such that $x_i \overset{\mathbf{n}}{\longleftrightarrow} y_i$ for every i. For every index i, fix a path γ_i from x_i to y_i (identified with the current $e \mapsto \mathbf{1}_{e \in \gamma_i}$) and define

$$\eta = \gamma_1 + \cdots + \gamma_k \mod 2.$$

We conclude the proof from the following facts. For every i, we have $\partial\gamma_i = \{x_i, y_i\}$, and for every currents $\mathbf{k}, \mathbf{m}$, the set of sources of the current $(\mathbf{k} + \mathbf{m} \mod 2)$ is equal to $\partial\mathbf{k} \Delta \partial\mathbf{m}$. □

Theorem 1.2 (Switching lemma)**.** *Let M, N be two independent ppp(J). For every $A, B, C \subset V$, we have*

$$\mathbb{P}[\partial M = A, \partial N = B, M + N \in \mathcal{F}_C] = \mathbb{P}[\partial M = A\Delta C, \partial N = B\Delta C, M + N \in \mathcal{F}_C].$$

Proof. We build a coupling between two pairs of independent ppp(J) (M, N) and $(\widetilde{M}, \widetilde{N})$ such that

$$\{\partial M = A, \partial N = B, M + N \in \mathcal{F}_C\} = \{\partial\widetilde{M} = A\Delta C, \partial\widetilde{N} = B\Delta C, \widetilde{M} + \widetilde{N} \in \mathcal{F}_C\}. \tag{3.1}$$

Let K be a ppp($2J$) on E. Independently, let $(Z_e^i)_{e \in E, i \ge 1}$ be iid Bernoulli(1/2) random variables.

If $K \in \mathcal{F}_C$, define $\eta = \eta(K) \in \{0,1\}^E$ such that $\eta \leq K$ and $\partial\eta = C$ (its existence is guaranteed by the third item of Lemma 1.1). Otherwise, set $\eta = \varnothing$. Define

$$\widetilde{Z}_e^i := \begin{cases} 1 - Z_e^i & \text{if } \eta_e = 1 \text{ and } i = 1, \\ Z_e^i & \text{otherwise.} \end{cases}$$

for every $e \in E$ and $i \geq 1$. One can check that $(\widetilde{Z}_e^i)_{e\in E, i\geq 1}$ are also iid Bernoulli (1/2) random variables, independent of K. By the thinning property of Poisson processes, M and N defined by

$$\forall e \in E \quad M_e = \sum_{i=1}^{K_e} Z_e^i \quad \text{and} \quad N_e = K_e - M_e$$

are two independent ppp(J). Equivalently, we define $\widetilde{M}$ and $\widetilde{N}$ by using $\widetilde{Z}$ instead of Z. Again $\widetilde{M}$ and $\widetilde{N}$ are two independent ppp(J), and they are related to M, N via

$$\widetilde{M}_e = M_e + \eta_e - 2\eta_e Z_e^1 \quad \widetilde{N}_e = M_e - \eta_e + 2\eta_e Z_e^1.$$

Therefore, if $K \in \mathcal{F}_C$, $\partial\widetilde{M}_e = \partial M_e \Delta \partial\eta$ and $\partial\widetilde{N}_e = \partial N_e \Delta \partial\eta$. Hence, for every $A, B \subset V$, we have

$$\{\partial M = A, \partial N = B, K \in \mathcal{F}_C\} = \{\partial\widetilde{M} = A\Delta C, \partial\widetilde{N} = B\Delta C, K \in \mathcal{F}_C\}.$$

Since the $K = M + N = \widetilde{M} + \widetilde{N}$, this proves Equation (3.1), and completes the proof. □

2 Probabilistic interpretation of the multi-point function

In this section, we intoduce a duplication method which, together with the switrching lemma, provides us with a probabilistic interpretation of $\langle\sigma_A\rangle^2$.

Duplication method The duplication method is a general idea to compute product of expectations of the form $\mathbb{E}[f(X)]\mathbb{E}[g(X)]$, where X is a general random variable, and f, g are two measurable map. By considering an independent copy Y with the same law as X, we can write

$$\mathbb{E}[f(X)]\mathbb{E}[g(X)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)] = \mathbb{E}[f(X)g(Y)],$$

provided all the expectations are well defined. This way, we transform the computation of a product of expectation into a single expectation.

Duplication and switching Let us apply the duplication method to random currents. Let $A \subset V$. Let M, N be two independent ppp(J) on E. We have

$$\langle\sigma_A\rangle\langle\sigma_A\rangle = \frac{\mathbb{P}[\partial M = A]}{\mathbb{P}[\partial M = \varnothing]} \cdot \frac{\mathbb{P}[\partial N = A]}{\mathbb{P}[\partial N = \varnothing]} = \frac{\mathbb{P}[\partial M = A, \partial N = A]}{\mathbb{P}[\partial M = \varnothing, \partial N = \varnothing]}. \tag{3.2}$$

This manipulation allows us to apply the switching lemma. First, we use Items 1 and 2 of Lemma 1.1 to introduce the event $\mathcal{F}_A$: on the event $\partial N = A$, we necessarily have $N \in \mathcal{F}_A$ and therefore $M + N \in \mathcal{F}_A$. Hence,

$$\begin{aligned}\mathbb{P}[\partial M = A, \partial N = A] &= \mathbb{P}[\partial M = A, \partial N = A, M + N \in \mathcal{F}_A] \\ &= \mathbb{P}[\partial M = \varnothing, \partial N = \varnothing, M + N \in \mathcal{F}_A],\end{aligned}$$

where we apply the switching lemma (Theorem 1.2) to $A = B = C$. By plugging this expression in (3.2), we finally obtain

$$\boxed{\langle\sigma_A\rangle^2 = \mathbb{P}[M + N \in \mathcal{F}_A \mid \partial M = \partial N = \varnothing]}.$$

In the particular case $A = xy$, we obtain

$$\boxed{\langle\sigma_x\sigma_y\rangle^2 = \mathbb{P}[x \xleftrightarrow{M+N} y \mid \partial M = \partial N = \varnothing]},$$

which gives a neat interpretation of the two-point function as the connection probability in the superposition of two independent sourceless currents.

3 Second GKS inequality

Theorem 3.1. *For every $A, B \subset V$ we have*

$$\langle\sigma_A\sigma_B\rangle \geq \langle\sigma_A\rangle\langle\sigma_B\rangle$$

Remark 3.2. *It implies the first GKS inequality. If A is even and non empty, consider $x \in A$ and apply the second GKS inequality to $A\backslash\{x\}$ and $\{x\}$.*

Proof. We first use the duplication principle and express the product $\langle\sigma_A\rangle\langle\sigma_B\rangle$ in terms of two independent random currents M and N, and then compare the two expressions by using the switching lemma.

$$\langle\sigma_A\rangle\langle\sigma_B\rangle = \frac{\mathbb{P}[\partial M = A]}{\mathbb{P}[\partial M = \varnothing]} \cdot \frac{\mathbb{P}[\partial N = B]}{\mathbb{P}[\partial N = \varnothing]} = \frac{\mathbb{P}[\partial M = A, \partial N = B]}{\mathbb{P}[\partial M = \varnothing, \partial N = \varnothing]}. \tag{3.3}$$

The quantity $\langle\sigma_A\sigma_B\rangle$ is not directly a product of two terms, but we can write it as $\langle\sigma_{A\Delta B}\rangle\langle\sigma_\varnothing\rangle$ to apply a duplication method and obtain the expression

$$\langle\sigma_A\sigma_B\rangle = \frac{\mathbb{P}[\partial M = A\Delta B, \partial N = \varnothing]}{\mathbb{P}[\partial M = \varnothing, \partial N = \varnothing]}. \tag{3.4}$$

In order to compare the numerators in Equations (3.3) and (3.4), we apply the switching lemma to switch the sources at B from one current to the other. To this aim, we first use Items 1 and 2 of Lemma 1.1 to introduce the event $\mathcal{F}_B$: on the event $\partial N = B$, we necessarily have $N \in \mathcal{F}_B$ and therefore $M + N \in \mathcal{F}_B$. Hence,

$$\begin{aligned}\mathbb{P}[\partial M = A, \partial N = B] &= \mathbb{P}[\partial M = A, \partial N = B, M + N \in \mathcal{F}_B] \\ &= \mathbb{P}[\partial M = A\Delta B, \partial N = \varnothing, M + N \in \mathcal{F}_B],\end{aligned}$$

where we used the switching lemma (Theorem 1.2) in the second line.

By plugging this expression in (3.3) and taking the difference with (3.4), we finally obtain

$$\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle = \frac{\mathbb{P}[\partial M = A\Delta B, \partial N = \varnothing, M + N \notin \mathcal{F}_B]}{\mathbb{P}[\partial M = \varnothing, \partial N = \varnothing]},$$

which concludes the proof. □

4 Monotonicity in the weights

In this section, we consider the Ising model with different weights. To avoid confusion, write μ_J for the Ising measure on G with weights J, $\langle \cdot \rangle_J$ for the corresponding expectation and Z_J for the partition function. Given some weights J and J' on G, write $J \leq J'$ if $J_e \leq J'_e$ for every $e \in E$.

Proposition 4.1 (monotonicity in J)**.** *Let J, J' be weights on G. For every $A \subset V$, we have*

$$J \leq J' \quad \Longrightarrow \quad \langle \sigma_A \rangle_J \leq \langle \sigma_A \rangle_{J'}$$

Proof. Defining $g(\sigma) = \exp\big(\sum_{e\in E}(J'_e - J_e)\sigma_e\big)$, we have

$$Z_{J'}[\sigma_A] = \sum_{\sigma\in\Omega} \sigma_A \exp\Big(\sum_{e\in E} J'_e \sigma_e\Big) = \sum_{\sigma\in\Omega} \sigma_A g(\sigma) \exp\Big(\sum_{e\in E} J_e \sigma_e\Big) = Z_J[\sigma_A g]$$

Hence,

$$\langle \sigma_A \rangle_{J'} = \frac{Z_J[\sigma_A g]}{Z_J[g]} = \frac{\langle g\sigma_A \rangle_J}{\langle g \rangle_J}. \tag{3.5}$$

By writing the exponential as a series and expanding all the products, we can rewrite g as

$$g(\sigma) = \sum_{k\in\mathbb{N}} \frac{1}{k!} \Big(\sum_{e\in E}(J'_e - J_e)\sigma_e\Big)^k = \sum_{S\subset V} \alpha_S \sigma_S,$$

with $\alpha_S \geq 0$ for every S. Therefore, by GKS inequality and linearity, we have $\langle g\sigma_A \rangle_J \geq \langle g \rangle_J \langle \sigma_A \rangle$ which, together with Equation (3.5) concludes $\langle \sigma_A \rangle_{J'} \geq \langle \sigma_A \rangle_J$.

□

5 Simon-Lieb inequality

Notation Let $S \subset V$. We write μ^S the Ising measure on the graph induced by S (i.e. the graph with vertex set S and edge set $\{e \in E : e \subset S\}$), and $\langle\cdot\rangle_S$ for the corresponding expectation. For $\mathbf{n} \in \mathbb{N}^E$ we write $\mathbf{n}^S$ for the current on G defined by

$$\mathbf{n}_e^S = \begin{cases} \mathbf{n}_e & \text{if } e \subset S \\ 0 & \text{otherwise.} \end{cases}$$

Notice that with this notation we have

$$\langle\sigma_A\rangle_S = \frac{\mathbb{P}[\partial N^S = A]}{\mathbb{P}[\partial N^S = \varnothing]}.$$

for every $A \subset S \subset V$.

The switching lemma of Section 1 has several generalizations: for example the two currents M and N may be defined on larger and possibly different graphs than the part where we switch the sources. The lemma below allows us to switch sources between M to N^S, provided the sources are in S

Lemma 5.1. *Let $S \subset V$. Let M, N be two independent ppp(J). For every $x, y, z \in V$ distinct, we have*

$$\mathbb{P}[\partial M_S = \varnothing, \partial N = xz, x \xleftrightarrow{M^S+N^S} y] = \mathbb{P}[\partial M_S = xy, \partial N = yz]$$

Proof. First decompose $N = N^S + N - N^S$, and notice that the two currents N^S and $N - N^S$ are independent (since they have disjoint supports). Therefore

$$\begin{aligned}\mathbb{P}[\partial M^S = \varnothing, \partial N = xz, x \xleftrightarrow{N^S} y] &= \sum_{\mathbf{k}\in\mathbb{N}^E} \mathbb{P}[\partial M^S = \varnothing, \partial N^S = \partial\mathbf{k}\Delta xz, x \xleftrightarrow{M^S+N^S} y]\mathbb{P}[N - N^S = \mathbf{k}] \\ &= \sum_{\mathbf{k}\in\mathbb{N}^E} \mathbb{P}[\partial M^S = xy, \partial N^S = \partial\mathbf{k}\Delta yz]\mathbb{P}[N - N^S = \mathbf{k}] \\ &= \mathbb{P}[\partial M^S = xy, \partial N = yz],\end{aligned}$$

where we apply the switching Lemma to (M^S, N^S) and $A = \varnothing, B = \partial\mathbf{k}\Delta xz, C = xy$ in the second line. □

Theorem 5.2 (Simon-Lieb inequality)**.** *Let $S \subset V$ and consider its inner vertex boundary $\partial_{\text{in}} S$, defined as the set of vertices of S with at least one neighbour outside S. For every $x \in S$ and $z \in V \backslash S$, we have*

$$\langle\sigma_x\sigma_z\rangle \le \sum_{y\in\partial_{\text{in}}S} \langle\sigma_x\sigma_y\rangle_S\langle\sigma_y\sigma_z\rangle$$

Proof. Without loss of generality, we may assume $x \notin \partial_{\text{in}} S$ (otherwise the inequality is trivially satisfied). We first express the two point function using the random current representation, and a duplication principle

$$\langle \sigma_x \sigma_z \rangle = \frac{\mathbb{P}[\partial N = xz]}{\mathbb{P}[\partial N = \varnothing]} = \frac{\mathbb{P}[\partial M^S = \varnothing, \partial N = xz]}{\mathbb{P}[\partial M^S = \varnothing, \partial N = \varnothing]}. \tag{3.6}$$

Recall that if N has two sources x and z, there must exist a path of $G[N > 0]$ from $x \in S$ to $z \in V \backslash S$. By considering the first portion of such path, before the moment it exits S, one can see that there must exist a vertex $y \in \partial_{\text{in}} S$ such that x is connected to y in $G[N^S > 0]$, and therefore in $G[M^S + N^S > 0]$. Therefore, by the union bound, we have

$$\begin{aligned}\mathbb{P}[\partial M^S = \varnothing, \partial N = xz] &\leq \sum_{y \in \partial_{\text{in}} S} \mathbb{P}[\partial M^S = \varnothing, \partial N = xz, x \xleftrightarrow{M^S + N^S} y] \\ &= \sum_{y \in \partial_{\text{in}} S} \mathbb{P}[\partial M_S = xy, \partial N = yz],\end{aligned}$$

where the last equality corresponds to Lemma 5.1. Plugging this inequality in the expression (3.6) concludes the proof.

$\square$