Mathematical Foundations for Finance

Chapter IV: Basics on Brownian Motion

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Definition and Properties

General (continuous-time) setting

We will work on a probability space $(\Omega, \mathcal{F}, P)$ which is *'rich enough for our purposes'*. In particular, Ω cannot be finite or countable.

We work with a *filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ in continuous time. As in discrete time, this is a family of σ-fields $\mathcal{F}_t \subseteq \mathcal{F}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. The time parameter runs either through $t \in [0, T]$ with a fixed time horizon $T \in (0, \infty)$ or through $t \in [0, \infty)$. In the latter case, we define

$$\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t := \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t \right).$$

For technical reasons, we assume (or make sure, if we construct the filtration) that $\mathbb{F}$ satisfies the so-called *'usual conditions'* of being *right-continuous and P-complete*.

Definition. A *Brownian motion with respect to P and a filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a (real-valued) stochastic process $W = (W_t)_{t \geq 0}$ which is adapted to $\mathbb{F}$, starts at 0 (i.e. $W_0 = 0$ P-a.s.) and satisfies:

(BM1) For $s \leq t$, the *increment* $W_t - W_s$ is independent (under P) of $\mathcal{F}_s$ with (under P) a normal distribution $\mathcal{N}(0, t - s)$.

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Brownian motion in $\mathbb{R}^m$ is simply an adapted $\mathbb{R}^m$-valued stochastic process null at 0 with (BM2) and such that (BM1) holds with $\mathcal{N}(0, t-s)$ replaced by $\mathcal{N}(0, (t-s)I_{m \times m})$, where $I_{m \times m}$ denotes the $m \times m$ identity matrix.

Brownian Motion - remarks

- One can prove that Brownian motion exists. See the course on "Brownian Motion and Stochastic Calculus" (in short BMSC).
- The letter W is used in honour of Norbert Wiener who gave the first rigorous proof of the existence of Brownian motion in 1923.
- However, Brownian motion was used considerably earlier in both finance and physics — by Louis Bachelier in his PhD thesis in 1900 for finance and by Albert Einstein in 1905 for physics.

BM - alternative definition

There is also a definition of Brownian motion without any filtration $\mathbb{F}$:

This is a (real-valued) stochastic process $W = (W_t)_{t \geq 0}$ which starts at 0, satisfies (BM2) and instead of (BM1) the following property:

(BM1′) For any $n \in \mathbb{N}$ and any times $0 = t_0 < t_1 < \cdots < t_n < \infty$, the increments $W_{t_i} - W_{t_{i-1}}$, $i = 1, \ldots, n$, are independent (under P) and we have (under P) that $W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})$, or $\mathcal{N}(0, (t_i - t_{i-1}) I_{m \times m})$ if W is $\mathbb{R}^m$-valued.

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The two definitions of BM are equivalent if one chooses as $\mathbb{F}$ the filtration $\mathbb{F}^W$ generated by W (and made right-continuous and P-complete).

More details can be found in the lecture notes on "Brownian Motion and Stochastic Calculus".

Proposition. Suppose $W = (W_t)_{t\geq 0}$ is a BM. Then:

1. $W^1 := -W$ is a BM.

Brownian Motion - First Properties

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5. The process W^5_t, $t \geq 0$, defined by

$$W^5_t := \begin{cases} tW_{\frac{1}{t}} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

is a BM *(inversion of small and large times)*.

Proposition. Suppose $W = (W_t)_{t \geq 0}$ is a BM. Then:

1. *Law of large numbers:* BM grows more slowly than linearly as $t \to \infty$,

$$\lim_{t \to \infty} \frac{W_t}{t} = 0 \quad P\text{-a.s.}$$

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that means for P-almost all ω, the function $t \mapsto W_t(\omega)$ for $t \to \infty$ oscillates precisely between $t \mapsto \pm\psi_{\mathrm{glob}}(t)$.

3. *(Local) Law of the iterated logarithm (LIL):* With $\psi_{\mathrm{loc}}(h) := \sqrt{2h \log(\log \frac{1}{h})}$, we have for every $t \geq 0$, P-a.s.

$$\liminf_{h \searrow 0} \frac{W_{t+h} - W_t}{\psi_{\mathrm{loc}}(h)} = -1, \qquad \limsup_{h \searrow 0} \frac{W_{t+h} - W_t}{\psi_{\mathrm{loc}}(h)} = +1$$

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i.e. for P-almost all ω, to the right of t, the trajectory $u \mapsto W_u(\omega)$ around the level $W_t(\omega)$ oscillates precisely between $h \mapsto \pm\psi_{\mathrm{loc}}(h)$.

Brownian Motion - Remarks

- One consequence of 2. and 3. is that BM crosses any level a infinitely many times — and once it is at that level, it even manages to achieve these infinitely many crossings in an arbitrarily short amount of time. This is a first indication of the amazingly *strong activity* of BM.

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- Part 1. of the proposition is easily proved by using part 5. of the previous proposition.
- Part 2. follows directly from part 3. via part 5. of the previous proposition.
- To show part 3., it is enough to take $t = 0$, by part 2. of the previous proposition, and to prove the $\limsup$ result, by part 1. of the previous proposition. But then the easy reductions stop and the proof becomes difficult.

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Brownian Motion - Differentiability?

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Proposition. Suppose $W = (W_t)_{t \geq 0}$ is a BM. Then for P-almost all $\omega \in \Omega$, the function $t \mapsto W_t(\omega)$ from $[0, \infty)$ to $\mathbb{R}$ is continuous, but nowhere differentiable.

The deeper reason behind the wild behaviour of Brownian trajectories, and the key to understanding stochastic calculus and Itô's formula for BM, is that Brownian trajectories are **continuous functions with non-zero quadratic variation**.

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Heuristically, this can be seen as follows:

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- So, if we want to approximate it with a 2point distribution, roughly

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- In very loose terms, this means that infinitesimal increments "$dW_t = W_{t+dt} - W_t$" of BM have the property that

$$\text{“}(dW_t)^2 = dt\text{”}.$$

A more precise description is as follows.

Definition Call a *partition* of $[0, \infty)$ any set $\Pi \subseteq [0, \infty)$ of time points with $0 \in \Pi$ and $\Pi \cap [0, T]$ finite for all $T \in [0, \infty)$.

This implies that Π is at most countable and can be ordered increasingly as $\Pi = \{0 = t_0 < t_1 < \cdots < t_m < \cdots < \infty\}$.

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- For any partition Π of $[0, \infty)$, any function $g : [0, \infty) \to \mathbb{R}$ and any $p > 0$, we first define the *p-variation of g on $[0, T]$ along Π* as

$$V_T^p(g, \Pi) := \sum_{t_i \in \Pi} |g(t_i \wedge T) - g(t_{i-1} \wedge T)|^p.$$

p-Variation - definition

One can then define the *p-variation of g on $[0, T]$* as

$$V_T^p(g) := \sup_{\Pi} V_T^p(g, \Pi),$$

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- For a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions of $[0, \infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$, one can also define the *p-variation of g on $[0, T]$ along $(\Pi_n)_{n\in\mathbb{N}}$* as

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provided that the limit exists.

- With the above notations, a function g is of *finite variation* or has *finite 1-variation* if $V_T^1(g) < \infty$ for every $T \in (0, \infty)$.

The interpretation is that the *graph of g has finite length on any time interval*.

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More precisely, define the *arc length* of (the graph of) g on the interval $[0, T]$ as

$$\sup_{\Pi} \sum_{t_i \in \Pi} \sqrt{(t_i \wedge T - t_{i-1} \wedge T)^2 + \big(g(t_i \wedge T) - g(t_{i-1} \wedge T)\big)^2},$$

with the supremum again taken over all partitions Π of $[0, \infty)$. Then g has finite variation on $[0, T]$ if and only if it has finite arc length on $[0, T]$.

This can be checked by using the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$.

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Moreover, one can show that any function of finite variation can be written as the difference of two increasing functions (and vice versa).

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Let us return to Brownian motion, taking $p = 2$ and as g one trajectory $W_{\cdot}(\omega)$. Then

$$Q_T^{\Pi} := \sum_{t_i \in \Pi} \left(W_{t_i \wedge T} - W_{t_{i-1} \wedge T} \right)^2 = V_T^2(W_{\cdot}, \Pi)$$

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is the *sum up to time T of the squared increments of BM along Π*.

With the above formal intuition "$(dW_t)^2 = dt$", we then expect, at least for $|\Pi|$ very small that

$$(W_{t_i \wedge T} - W_{t_{i-1} \wedge T})^2 \approx t_i \wedge T - t_{i-1} \wedge T$$

and hence

$$Q_T^\Pi \approx \sum_{t_i \in \Pi} (t_i \wedge T - t_{i-1} \wedge T) = T \qquad \text{for } |\Pi| \text{ small.}$$

Even if the previous reasoning is only heuristic, the result is correct:

Theorem. Let $W = (W_t)_{t \geq 0}$ be a BM. For any sequence $(\Pi_n)_{n \in \mathbb{N}}$ of partitions of $[0, \infty)$ which is refining (i.e. $\Pi_n \subseteq \Pi_{n+1}$ for all n) and satisfies $\lim_{n \to \infty} |\Pi_n| = 0$, we have

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We express this by saying that along $(\Pi_n)_{n \in \mathbb{N}}$, the Brownian motion W has (with probability 1) *quadratic variation t on $[0, t]$ for every $t \geq 0$*, and we write $\langle W \rangle_t = t$.

We sometimes also say that P-almost all trajectories $W_\cdot(\omega) : [0, \infty) \to \mathbb{R}$ of BM have quadratic variation t on $[0, t]$, for each $t \geq 0$.

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   More generally, if $0 < p < q$, then

   - $\lim_{n\to\infty} V_T^q(f, \Pi_n) > 0$ implies that $\lim_{n\to\infty} V_T^p(f, \Pi_n) = +\infty$,
   - while $\lim_{n\to\infty} V_T^p(f, \Pi_n) < \infty$ implies that $\lim_{n\to\infty} V_T^q(f, \Pi_n) = 0$.

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2. In the previous theorem, it is important that the partitions Π_n do not depend on the trajectory $W_\cdot(\omega)$, but are fixed a priori.

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   - the convergence holds not P-almost surely, but only in probability,
   - the map $t \mapsto [M]_t(\omega)$ is continuous only if M itself has continuous trajectories.

Martingale properties and results

There are many martingales which are naturally associated to Brownian motion, and this is useful in many different contexts. We present here just a small sample that will be used or useful later.

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As in discrete time:

Definition. A *martingale* with respect to P and $\mathbb{F}$ is a (real-valued) stochastic process $M = (M_t)$ such that

- M is adapted to $\mathbb{F}$,
- M is P-integrable in the sense that each M_t is in $L^1(P)$,
- and the *martingale property* holds: for $s \leq t$, we have

$$E[M_t \,|\, \mathcal{F}_s] = M_s \qquad P\text{-a.s.} \tag{1}$$

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If we have in (1) the inequality "$\leq$" instead of "$=$", then M is a *supermartingale*; if we have "$\geq$", then M is a *submartingale*. Of course, $\mathbb{F} = (\mathcal{F}_t)$ and $M = (M_t)$ should have the same time index set.

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We can and do therefore always assume that our martingales have nice trajectories in that sense, and this is important for some of the subsequent results. We shall point this out more explicitly when it is used.

As in discrete time:

Definition. A *stopping time* with respect to $\mathbb{F}$ is a mapping $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Stopping times in continuous time

As in discrete time:

Definition. A *stopping time* with respect to $\mathbb{F}$ is a mapping $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

A standard example is the first time that some adapted right-continuous process X (e.g. BM W) hits an open set B (e.g. (a, ∞)), that is

$$\begin{aligned}\tau &:= \inf\{t \geq 0 \,|\, X_t \in B\} \\ &\Big(= \inf\{t \geq 0 \,|\, W_t > a\}, \text{ for } X = W \text{ and } B = (a, \infty)\Big).\end{aligned}$$

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- Checking the stopping time property above uses the right-continuity of the filtration.
- τ above is still a stopping time if B is allowed to be any Borel set, but the proof of this apparently minor extension is surprisingly difficult.

One of the most useful properties of martingales is that the martingale property (1) and its consequences very often extend to the case where the fixed times $s \leq t$ are replaced by stopping times $\sigma \leq \tau$.

"Very often" means under additional conditions, as we shall see presently.

One of the most useful properties of martingales is that the martingale property (1) and its consequences very often extend to the case where the fixed times $s \leq t$ are replaced by stopping times $\sigma \leq \tau$.

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Definition.

- To make sense of (1) for σ and τ, we first need to define, for a stopping time σ, the σ-field of *events observable up to time σ* as

$$\mathcal{F}_\sigma := \left\{ A \in \mathcal{F} \,\middle|\, A \cap \{\sigma \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0 \right\}.$$

(One must and can check that $\mathcal{F}_\sigma$ is a σ-field, and one has $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$ for $\sigma \leq \tau$.)

- Then we define *M_τ, the value of M at the stopping time τ*, by

$$(M_\tau)(\omega) := M_{\tau(\omega)}(\omega).$$

Note that this implicitly assumes that we have a random variable M_∞, because τ can take the value $+\infty$.

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- One can then prove that if τ is a stopping time and M is an adapted process with RC trajectories, then *M_τ is $\mathcal{F}_\tau$-measurable* (as one intuitively expects).

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- One can then prove that if τ is a stopping time and M is an adapted process with RC trajectories, then *M_τ is $\mathcal{F}_\tau$-measurable* (as one intuitively expects).

- Finally, we recall the *stopped process $M^\tau = (M_t^\tau)_{t\geq 0}$* which is defined by $M_t^\tau := M_{t\wedge\tau}$ for all $t \geq 0$. If M is adapted with RC trajectories and τ is a stopping time, then also M^τ is adapted and has RC trajectories.

After the above preliminaries, we now have

Theorem. (Stopping theorem) Suppose that $M = (M_t)_{t \geq 0}$ is a $(P, \mathbb{F})$-martingale with RC trajectories, and σ, τ are $\mathbb{F}$-stopping times with $\sigma \leq \tau$. If either τ is bounded by some $T \in (0, \infty)$ or M is uniformly integrable, then M_τ, M_σ are both in $L^1(P)$ and

$$E[M_\tau \,|\, \mathcal{F}_\sigma] = M_\sigma \qquad P\text{-a.s.} \tag{2}$$

Two frequent *applications* of the last theorem are the following:

1. For any RC martingale M and any stopping time τ, we have

$$E[M_{\tau\wedge t}\,|\,\mathcal{F}_s] = M_{\tau\wedge s} \text{ for } s \leq t \ ,$$

   that is the stopped process $M^\tau = (M^\tau_t)_{t\geq 0} = (M_{t\wedge\tau})_{t\geq 0}$ is again a martingale (because we have $E[M^\tau_t\,|\,\mathcal{F}_s] = M^\tau_s$).
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   [Because not necessarily $s \leq \tau \wedge t$, this needs a little bit of extra work.]

2. If M is an RC martingale and τ is any stopping time, then we always have for any $t \geq 0$ that $E[M_{\tau\wedge t}] = E[M_0]$. If either τ is bounded or M is uniformly integrable, then we also obtain $E[M_\tau] = E[M_0]$.

For future use, let us also recall the notion of a local martingale null at 0, now in continuous time.

Definition. An adapted process $X = (X_t)_{t\geq 0}$ null at 0 (i.e. with $X_0 = 0$) is called a *local martingale null at 0* (with respect to P and $\mathbb{F}$) if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to ∞ such that for each $n \in \mathbb{N}$, the stopped process $X^{\tau_n} = (X_{t\wedge\tau_n})_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale.

We then call $(\tau_n)_{n\in\mathbb{N}}$ a *localising sequence*.

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Proposition. Suppose $W = (W_t)_{t \geq 0}$ is a $(P, \mathbb{F})$-Brownian motion. Then the following processes are all $(P, \mathbb{F})$-martingales:

1. W itself.
2. $W_t^2 - t$, $t \geq 0$.
3. $e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$, $t \geq 0$, for any $\alpha \in \mathbb{R}$.

We do this argument also because it illustrates how to work with the properties of BM:

Proof: For each of the above processes, adaptedness is obvious, and integrability is also clear because each W_t has a normal distribution and hence all exponential moments.

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1. As $W_t - W_s$ is independent of $\mathcal{F}_s$ and $\sim \mathcal{N}(0, t-s)$, we get 1. from

$$E[W_t - W_s \,|\, \mathcal{F}_s] = E[W_t - W_s] = 0.$$

2. Using this with $W_t^2 - W_s^2 = (W_t - W_s)^2 + 2W_s(W_t - W_s)$ and $\mathcal{F}_s$-measurability of W_s then gives

$$E\big[W_t^2 - W_s^2 \,\big|\, \mathcal{F}_s\big] = E\big[(W_t - W_s)^2 \,\big|\, \mathcal{F}_s\big]$$
$$= E\big[(W_t - W_s)^2\big] = \mathrm{Var}[W_t - W_s] = t - s.$$

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3. Finally, setting $M_t := e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$ yields

$$E\left[\frac{M_t}{M_s} \,\middle|\, \mathcal{F}_s\right] = E\left[e^{\alpha(W_t - W_s) - \frac{1}{2}\alpha^2(t-s)} \,\middle|\, \mathcal{F}_s\right]$$
$$= e^{-\frac{1}{2}\alpha^2(t-s)} E\left[e^{\alpha(W_t - W_s)}\right] = 1$$

because $E[e^Z] = e^{\mu + \frac{1}{2}\sigma^2}$ for $Z \sim \mathcal{N}(\mu, \sigma^2)$. So we have 3. as well.

q.e.d.

Example. To illustrate that the conditions in the last theorem are really needed, consider a Brownian motion W and the stopping time

$$\tau := \inf\{t \geq 0 \,|\, W_t > 1\}.$$

Due to the law of the iterated logarithm, we have $\tau < \infty$ P-a.s., and because W has continuous trajectories, we get $W_\tau = 1$ P-a.s.

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For $\sigma = 0$, if (2) were valid for W and τ, σ, we should get by taking expectations that

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which is clearly false.

So τ *cannot be bounded* by a constant (in fact, one can even show that $E[\tau] = +\infty$), and W is a martingale, but *not uniformly integrable*.

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$$\tau_a := \inf\{t \geq 0 \,|\, W_t > a\},$$
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Note that $\tau_a < \infty$ P-a.s. by the (global) law of the iterated logarithm, whereas $\sigma_{a,b}$ can be $+\infty$ with positive probability (see below).

Proposition. Let W be a BM and $a > 0$, $b > 0$. Then for any $\lambda > 0$, we have

$$E\left[e^{-\lambda\tau_a}\right] = e^{-a\sqrt{2\lambda}} \tag{3}$$

and

$$E\left[e^{-\lambda\sigma_{a,b}}\right] = E\left[e^{-\lambda\sigma_{a,b}} I_{\{\sigma_{a,b}<\infty\}}\right] = e^{-a(b+\sqrt{b^2+2\lambda})}. \tag{4}$$

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Proof: We give this argument because it illustrates how to use the preceding martingale results. First of all, take $\alpha > 0$ and define

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Then M is a martingale, and hence so is the stopped process M^τ, for $\tau \in \{\tau_a, \sigma_{a,b}\}$.

This implies that

$$1 = E[M_0] = E[M_{\tau\wedge t}] = E\left[e^{\alpha W_{\tau\wedge t} - \frac{1}{2}\alpha^2(\tau\wedge t)}\right]$$

for all t, and we now want to let $t \to \infty$.

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For $\tau = \tau_a$, we have $W_{\tau_a\wedge t} \leq a$ and therefore $M_{\tau_a\wedge t}$ is bounded uniformly in t and ω (by $e^{\alpha a}$); so dominated convergence yields for $t \to \infty$ that

$$\begin{aligned} 1 &= \lim_{t\to\infty} E\left[e^{\alpha W_{\tau_a\wedge t} - \frac{1}{2}\alpha^2(\tau_a\wedge t)}\right] = E\left[\lim_{t\to\infty} e^{\alpha W_{\tau_a\wedge t} - \frac{1}{2}\alpha^2(\tau_a\wedge t)}\right] \\ &= E\left[e^{\alpha W_{\tau_a} - \frac{1}{2}\alpha^2\tau_a}\right] = e^{\alpha a} E\left[e^{-\frac{1}{2}\alpha^2\tau_a}\right] \end{aligned}$$

because $\tau_a < \infty$ P-a.s., and so $\alpha := \sqrt{2\lambda}$ gives (3).

For $\tau = \sigma_{a,b}$, we have $W_{\sigma_{a,b} \wedge t} \leq a + b(\sigma_{a,b} \wedge t)$ so that

$$M_{\sigma_{a,b} \wedge t} \leq \exp\left(\alpha a + \left(\alpha b - \frac{1}{2}\alpha^2\right)(\sigma_{a,b} \wedge t)\right)$$

is bounded uniformly in t and ω (by $e^{\alpha a}$) for $\alpha b < \frac{1}{2}\alpha^2$, i.e. for $\alpha > 2b$.

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is bounded uniformly in t and ω (by $e^{\alpha a}$) for $\alpha b < \frac{1}{2}\alpha^2$, i.e. for $\alpha > 2b$. Then we get in the same way as above via dominated convergence that

$$1 = e^{\alpha a} E\left[e^{(\alpha b - \frac{1}{2}\alpha^2)\sigma_{a,b}} I_{\{\sigma_{a,b}<\infty\}}\right] = e^{\alpha a} E\left[e^{(\alpha b - \frac{1}{2}\alpha^2)\sigma_{a,b}}\right],$$

using that $\alpha b - \frac{1}{2}\alpha^2 < 0$. So that, on the set $\{\sigma_{a,b} = +\infty\}$, $e^{(\alpha b - \frac{1}{2}\alpha^2)\sigma_{a,b}} = 0$ and $M_{\sigma_{a,b}\wedge t} \to 0$ as $t \to \infty$.

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Then (4) follows for $\alpha := b + \sqrt{b^2 + 2\lambda}$, which gives by a straightforward computation that $\alpha b - \frac{1}{2}\alpha^2 = \alpha(b - \frac{1}{2}\alpha) = -\lambda < 0$. **q.e.d.**

Remarks

Remark. If we let $\lambda \searrow 0$ in (4), we obtain $P[\sigma_{a,b} < \infty] = e^{-2ab}$, so that indeed $P[\sigma_{a,b} = +\infty] = 1 - e^{-2ab} > 0$.

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For a general random variable $U \geq 0$, the function $\lambda \mapsto E[e^{-\lambda U}]$ for $\lambda > 0$ is called the *Laplace transform* of U. Its general importance in probability theory is that it uniquely determines the distribution of U.

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In mathematical finance, both τ_a and $\sigma_{a,b}$ come up in connection with a number of so-called *exotic options*. In particular, they are important for *barrier options* whose payoff depends on whether or not a (upper or lower) level has been reached by a given time.

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When computing prices of such options in the Black–Scholes model, one almost immediately encounters the Laplace transforms in the previous proposition. For more details, see for instance Chapter 9 of Dana/Jeanblanc (2003).

Markovian Properties

We have already seen that for any fixed time $T \in (0, \infty)$, the process

$$W_{t+T} - W_T,\ t \geq 0, \qquad \text{is again a BM} \tag{5}$$

if $(W_t)_{t \geq 0}$ is a Brownian motion.

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if $(W_t)_{t \geq 0}$ is a Brownian motion.

This means that if we restart a BM from level 0 at some fixed time, it behaves exactly as if it had only just started. Moreover, one can show that the independence of increments of BM implies that

$$W_{t+T} - W_T,\ t \geq 0, \qquad \text{is independent of } \mathcal{F}^0_T, \tag{6}$$

where $\mathcal{F}^0_T = \sigma(W_s; s \leq T)$ is the σ-field generated by BM up to time T.

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where $\mathcal{F}_T^0 = \sigma(W_s; s \leq T)$ is the σ-field generated by BM up to time T.

Intuitively, this means that BM at any fixed time T simply *forgets its past up to time T and starts afresh*.

One consequence of (5) and (6) is the following:

Suppose that at some fixed time T, we are interested in the behaviour of W after time T and try to predict this on the basis of the past of W up to time T, where "prediction" is done in the sense of a conditional expectation.

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Then we may as well forget about the past and look only at the current value W_T at time T.

A bit more precisely, we can express this, for functions $g \geq 0$ applied to the part of BM after time T, as

$$E\Big[g(W_u; u \geq T) \,\Big|\, \sigma(W_s; s \leq T)\Big] = E\Big[g(W_u; u \geq T) \,\Big|\, \sigma(W_T)\Big]. \tag{7}$$

This is called the *Markov property* of BM, and it is very useful in many situations.

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Note, however, that quite apart from the difficulties of writing down an analogue of (7) for a random time $\tau(\omega)$, it is even not clear whether this should then be true, because after all, τ itself can explicitly depend on the past behaviour of BM.

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Note, however, that quite apart from the difficulties of writing down an analogue of (7) for a random time $\tau(\omega)$, it is even not clear whether this should then be true, because after all, τ itself can explicitly depend on the past behaviour of BM.

Nevertheless, it turns out that such a result is true; one says that BM even has the *strong Markov property*.

If we denote by $\mathbb{F}^W$ the filtration generated by W (and made right-continuous), and if τ is a stopping time with respect to $\mathbb{F}^W$ and such that $\tau < \infty$ P-a.s., then

$$W_{t+\tau} - W_\tau,\ t \geq 0, \qquad \text{is again a BM and independent of } \mathcal{F}_\tau^W.$$

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Of course, this includes (5) and (6) as a special case, and one can easily believe that it is even more useful than (7).

However, the proof is too difficult to be given here.