Mathematical Foundations for Finance

Chapter VI: Stochastic Calculus

Prof. Dr. Beatrice Acciaio
beatrice.acciaio@math.ethz.ch

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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Throughout this chapter, we fix the following setting:

- We work on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)$ satisfying the usual conditions of right-continuity and P-completeness
- For all local martingales, we then can (and do) choose a version with RCLL trajectories
- For the time t, we have:
  - either $t \in [0, T]$, with a fixed time horizon $T \in (0, \infty)$,
  - or $t \geq 0$
- In the latter case, we set

$$\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t := \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t\right)$$

Itô's Formula

In this section, we investigate the following questions:

1) If X is a semimartingale and f is some (suitable) function, what can we say about $f(X)$?
2) What kind of process is it, and what does it look like in more detail?

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1) If X is a semimartingale and f is some (suitable) function, what can we say about $f(X)$?
2) What kind of process is it, and what does it look like in more detail?

If $x : [0, \infty) \to \mathbb{R}$ is a function $t \mapsto x(t)$ and we think of x as a typical trajectory $t \mapsto X_t(\omega)$ of X, then the *classical chain rule from analysis* says:

- If x is in C^1 (i.e. continuously differentiable) and $f : \mathbb{R} \to \mathbb{R}$ is in C^1, the composition $f \circ x : [0, \infty) \to \mathbb{R}$, $t \mapsto f(x(t))$ is again in C^1 and its derivative is given by

$$\frac{d}{dt}(f \circ x)(t) = \frac{df}{dx}\big(x(t)\big)\,\frac{dx}{dt}(t)$$

- In differential notation, we can rewrite this as

$$d(f \circ x)(t) = f'\big(x(t)\big)\, dx(t),$$

or in integral form

$$f\big(x(t)\big) - f\big(x(0)\big) = \int_0^t f'\big(x(s)\big)\, dx(s).$$

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However:

- This classical result does not help us a lot in a stochastic context
- X might have only RCLL instead of continuous trajectories, and even if continuous, we cannot hope that they are of finite variation

- Recall that a *semimartingale* is a stochastic process of the form $X = X_0 + M + A$, where $M_0 = A_0 = 0$, M is a local martingale and A is an adapted process with RCLL trajectories of finite variation.

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- For any such A, the *quadratic variation* (along any fixed, i.e. nonrandom, sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions of $[0,\infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$) is given by the sum of the squared jumps of A, i.e.

$$[A]_t = \sum_{0<s\leq t} (\Delta A_s)^2 = \sum_{0<s\leq t} (A_s - A_{s-})^2 \qquad \text{for } t \geq 0.$$

Itô's Formula - Preparation

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- Then, by polarisation, we have for any semimartingale Y that

$$[A, Y]_t = \sum_{0<s\leq t} \Delta A_s \Delta Y_s \qquad \text{for } t \geq 0.$$

- The quadratic variation of a general semimartingale $X = X_0 + M + A$ has the form

$$
\begin{aligned}
[X] = [M + A] &= [M] + [A] + 2[M, A] \\
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- In particular, if A is continuous, we obtain that $[X] = [M]$, even if X (hence M) is only RCLL.

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- Then the following result can be shown:

  Any continuous function of finite variation has zero quadratic variation along any sequence $(\Pi_n)_{n\in\mathbb{N}}$ *of partitions of* $[0,\infty)$ *whose mesh size* $|\Pi_n|$ *goes to 0 as* $n\to\infty$.

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- So if the semimartingale X is continuous, then its (unique) finite variation part A has zero quadratic variation, and its (unique) local martingale part M has quadratic variation $[M] = \langle M \rangle$.
  Then the covariation of M and A is also zero by Cauchy–Schwarz, and $[X] = \langle X \rangle = [M] = \langle M \rangle$ which is again continuous.

Let us return to the transformation $f(X)$ of a semimartingale X by a function f. In the simplest case, the answer to our basic question in this section looks as follows.

Theorem. (Itô's formula I) *Suppose $X = (X_t)_{t\geq 0}$ is a continuous real-valued semimartingale and $f : \mathbb{R} \to \mathbb{R}$ is in C^2. Then $f(X) = (f(X_t))_{t\geq 0}$ is again a continuous (real-valued) semimartingale,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s \qquad P\text{-a.s.}$$

for all $t \geq 0$.

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2) The dX-integral is a *stochastic integral*. It is well defined as X is a semimartingale and $f'(X)$ is adapted and continuous, hence predictable and locally bounded.
3) The $d\langle X \rangle$-integral is a classical *Lebesgue–Stieltjes integral* as $\langle X \rangle$ has increasing trajectories. It is also well defined as $f''(X)$ is also predictable and locally bounded.

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3) The $d\langle X\rangle$-integral is a classical *Lebesgue–Stieltjes integral* as $\langle X\rangle$ has increasing trajectories. It is also well defined as $f''(X)$ is also predictable and locally bounded.
4) In purely formal *differential notation*, the last theorem is usually written more compactly as

$$\begin{aligned} df(X_t) &= f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,d\langle X\rangle_t \\ &= f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,d\langle M\rangle_t, \end{aligned}$$

using that $\langle X\rangle = \langle M\rangle$.

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6) One can view Itô's formula and its proof as a purely analytical result which provides an *extension of the chain rule* for $f \circ x$ to functions x that have a nonzero quadratic variation. This has been developed by Hans Föllmer.
7) To see the financial relevance of Itô's formula, think of X as some underlying financial asset and of $Y = f(X)$ as a new product obtained from the underlying by a possibly nonlinear transformation f (derivative written on X). Then Itô's formula shows us how the product reacts to changes in the underlying.

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Important: When using stochastic models (for X), a simple linear approximation is not good enough and one must also account for the second-order behaviour of X.

The easiest way to remember both the result and its proof is via the following rough argument:

A Taylor expansion (stopped at second order) at the infinitesimal level gives

$$df(X_t) = f(X_{t+dt}) - f(X_t) = f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)(dX_t)^2,$$

with $(dX_t)^2 = (X_{t+dt} - X_t)^2 = \langle X\rangle_{t+dt} - \langle X\rangle_t = d\langle X\rangle_t$.

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Note, however, that this reasoning is purely formal and does not constitute a correct proof. (For example, it does not explain why we stop at the second and not at the third order in the expansion.)

To show how we can make the above idea rigorous, we write for non-infinitesimal increments

$$f(X_{t_{i+1}\wedge t}) - f(X_{t_i}) = f'(X_{t_i})\big(X_{t_{i+1}\wedge t} - X_{t_i}\big) + \frac{1}{2}f''(X_{t_i})\big(X_{t_{i+1}\wedge t} - X_{t_i}\big)^2 + R_i,$$

where R_i stands for the error term in the Taylor expansion and the t_i come from a partition Π_n of $[0,\infty)$.

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where R_i stands for the error term in the Taylor expansion and the t_i come from a partition Π_n of $[0, \infty)$.

Now we sum over the $t_i \leq t$ and obtain on the left-hand side a telescoping sum which equals $f(X_t) - f(X_0)$. On the right-hand side, we have the convergence

$$Q_t^{\Pi_n} = \sum_{t_i \in \Pi_n,\, t_i \leq t} \big(X_{t_{i+1}\wedge t} - X_{t_i}\big)^2 \longrightarrow \langle X \rangle_t \qquad \text{as } |\Pi_n| \to 0.$$

This implies firstly by a weak convergence argument that

$$\frac{1}{2} \sum_{t_i \in \Pi_n,\, t_i \leq t} f''(X_{t_i})(X_{t_{i+1} \wedge t} - X_{t_i})^2 \longrightarrow \frac{1}{2} \int_0^t f''(X_s)\, d\langle X \rangle_s,$$

and secondly by a careful estimate that

$$\sum_{t_i \in \Pi_n,\, t_i \leq t} |R_i| \longrightarrow 0.$$

(This is exactly the point where the mathematical analysis shows why the second order is the correct order of expansion.)

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As a consequence, the sums

$$\sum_{t_i\in\Pi_n,\, t_i\le t} f'(X_{t_i})\big(X_{t_{i+1}\wedge t}-X_{t_i}\big)$$

must also converge, and the dominated convergence theorem for stochastic integrals implies that the limit is $\int_0^t f'(X_s)\,dX_s$. **q.e.d.**

Itô's Formula - Example

For $X = W$ a Brownian motion and $f(x) = x^2$, we obtain $f'(x) = 2x$, $f''(x) \equiv 2$ and therefore

$$W_t^2 = W_0^2 + \int_0^t 2W_s \, dW_s + \frac{1}{2} \int_0^t 2 \, d\langle W \rangle_s.$$

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Using $W_0 = 0$ and the fact that BM has quadratic variation $\langle W \rangle_t = t$, hence $d\langle W \rangle_s = ds$, gives

$$W_t^2 = 2 \int_0^t W_s \, dW_s + \int_0^t \, ds = 2 \int_0^t W_s \, dW_s + t$$

or rewritten

$$\int_0^t W_s \, dW_s = \frac{1}{2} W_t^2 - \frac{1}{2} t.$$

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Theorem. *Suppose $X = (X_t)_{t\geq 0}$ is a general $\mathbb{R}^d$-valued semimartingale and $f : \mathbb{R}^d \to \mathbb{R}$ is in C^2. Then $f(X) = (f(X_t))_{t\geq 0}$ is again a (real-valued) semimartingale, and we explicitly have P-a.s. for all $t \geq 0$*

1) *if X has continuous trajectories:*

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x^i}(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j \rangle_s,$$

or in more compact notation, using subscripts to denote partial derivatives,

$$df(X_t) = \sum_{i=1}^{d} f_{x^i}(X_t)\, dX_t^i + \frac{1}{2} \sum_{i,j=1}^{d} f_{x^i x^j}(X_t)\, d\langle X^i, X^j \rangle_t.$$

Theorem. (Continued...)

2) if $d = 1$ (so that X is real-valued, but not necessarily continuous):

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})\, dX_s + \frac{1}{2}\int_0^t f''(X_{s-})\, d[X]_s + \sum_{0<s\leq t} \left(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s - \frac{1}{2} f''(X_{s-})(\Delta X_s)^2 \right).$$

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$$+ \sum_{0<s\le t}\Big(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s - \frac{1}{2}f''(X_{s-})(\Delta X_s)^2\Big).$$

There is of course also a version of Itô's formula for general $\mathbb{R}^d$-valued semimartingales (which contains both 1) and 2) as special cases). It looks similar to part 2) but of course has in addition sums like in part 1), with $\langle\cdot,\cdot\rangle$ replaced by $[\cdot,\cdot]$.

Example I

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Therefore we only need the derivatives

$$f_x = \frac{\partial f}{\partial x},\ f_t = \frac{\partial f}{\partial t},\ f_{xx} = \frac{\partial^2 f}{\partial x^2}.$$

and obtain that

$$dY_t = f_x(W_t, t)\, dW_t + \big(f_t + \frac{1}{2} f_{xx}\big)(W_t, t)\, dt \qquad \text{for } 0 \le t < T.$$

Example II

- The CRR binomial model can be written as

$$\begin{aligned}
\frac{\widetilde{S}^0_k - \widetilde{S}^0_{k-1}}{\widetilde{S}^0_{k-1}} &= r, \\
\frac{\widetilde{S}^1_k - \widetilde{S}^1_{k-1}}{\widetilde{S}^1_{k-1}} &= Y_k - 1 =: R_k = E[R_k] + (R_k - E[R_k]).
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- Passing from time steps of size 1 to dt and noting that Brownian increments have expectation 0 like the term $R_k - E[R_k]$, a continuous-time analogue would be of the form

$$\frac{d\widetilde{S}^0_t}{\widetilde{S}^0_t} = r\,dt,$$

$$\frac{d\widetilde{S}^1_t}{\widetilde{S}^1_t} = \mu\,dt + \sigma\,dW_t.$$

- Of course, the equation for $\widetilde{S}^0$ is just a very simple ordinary differential equation (ODE), whose solution for the starting value $\widetilde{S}^0_0 = 1$ is $\widetilde{S}^0_t = e^{rt}$.

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- The equation for $\widetilde{S}^1$ is a *stochastic differential equation (SDE)*, and its solution is given by the *geometric Brownian motion (GBM)*

$$\widetilde{S}^1_t = \widetilde{S}^1_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right) \qquad \text{for } t \geq 0.$$

Note the possibly surprising term $-\frac{1}{2}\sigma^2$.

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$$\widetilde{S}_t^1 = \widetilde{S}_0^1 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right) \qquad \text{for } t \geq 0.$$

Note the possibly surprising term $-\frac{1}{2}\sigma^2$.

- To see that this is indeed the solution, we write

$$\widetilde{S}_t^1 = f(W_t, t) \qquad \text{with } f(x, t) = \widetilde{S}_0^1 \, e^{\sigma x + (\mu - \frac{1}{2}\sigma^2)t}.$$

We now apply Itô's formula for $d = 2$ to $X_t = (W_t, t)$.

- As the second component $X_t^2 = t$ is continuous and increasing, it has finite variation. so we only need the derivatives

$$f_x = \frac{\partial f}{\partial x} = \sigma f,$$
$$f_t = \frac{\partial f}{\partial t} = \left(\mu - \frac{1}{2}\sigma^2\right) f,$$
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Example II (continued...)

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- Then we get, by using that $\langle W \rangle_t = t$ and $f(W_t, t) = \widetilde{S}_t^1$, that

$$\begin{aligned} d\widetilde{S}_t^1 &= f_x(W_t, t)\, dW_t + f_t(W_t, t)\, dt + \frac{1}{2} f_{xx}(W_t, t)\, d\langle W \rangle_t \\ &= \sigma \widetilde{S}_t^1\, dW_t + \left(\mu - \frac{1}{2}\sigma^2\right) \widetilde{S}_t^1\, dt + \frac{1}{2}\sigma^2 \widetilde{S}_t^1\, dt \\ &= \widetilde{S}_t^1(\sigma\, dW_t + \mu\, dt), \end{aligned}$$

exactly as claimed.

Example III

- If $X = (X_t)_{t \geq 0}$ is a continuous real-valued semimartingale, then

$$Z_t := e^{X_t - X_0 - \frac{1}{2}\langle X \rangle_t} \qquad \text{for } t \geq 0$$

is the unique solution of the SDE

$$dZ_t = Z_t \, dX_t, \qquad Z_0 = 1.$$

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is the unique solution of the SDE

$$dZ_t = Z_t \, dX_t, \qquad Z_0 = 1.$$

- Put differently, this means that Z satisfies

$$Z_t = 1 + \int_0^t Z_s \, dX_s \qquad \text{for all } t \geq 0, \ P\text{-a.s.}$$

Checking that the above Z does satisfy the above SDE, as well as proving uniqueness of the solution, is a good *exercise* in the use of Itô's formula.

The following definition

Definition. For a general real-valued semimartingale X null at 0, the *stochastic exponential* of X is defined as the unique solution Z of the SDE

$$dZ_t = Z_{t-}\, dX_t, \qquad Z_0 = 1,$$

i.e. $Z_t = 1 + \int_0^t Z_{s-}\, dX_s$ for all $t \geq 0$, and it is denoted by $\mathcal{E}(X) := Z$.

Example IV

- Suppose W is a Brownian motion, $T \in (0, \infty)$ is fixed and $h : \mathbb{R} \to \mathbb{R}$ is a measurable function with $h(W_T) \in L^1$. Then clearly

$$M_t := E[h(W_T) \,|\, \mathcal{F}_t] \qquad \text{for } 0 \le t \le T$$

is a martingale.

- Suppose W is a Brownian motion, $T \in (0,\infty)$ is fixed and $h : \mathbb{R} \to \mathbb{R}$ is a measurable function with $h(W_T) \in L^1$. Then clearly

$$M_t := E[h(W_T) \,|\, \mathcal{F}_t] \qquad \text{for } 0 \le t \le T$$

is a martingale.

- By writing

$$M_t = E\big[h(W_t + W_T - W_t) \,\big|\, \mathcal{F}_t\big]$$

and using that W_t is $\mathcal{F}_t$-measurable and $W_T - W_t$ is independent of $\mathcal{F}_t$ and $\sim \mathcal{N}(0, T-t)$, we get

$$M_t = E[h(x + W_T - W_t)]\big|_{x=W_t} = f(W_t, t)$$

with

$$f(x,t) = E[h(x + W_T - W_t)] = \int_{-\infty}^{\infty} h(x+y) \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{y^2}{2(T-t)}}\, dy.$$

Example IV (continued...)

- ▷ So $f(\cdot, t)$, as a function of x for fixed t, is the convolution of h with a function in C^∞ and therefore also C^∞ with respect to x, and $f(x, \cdot)$ is clearly in C^1 with respect to t as long as $t < T$.

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- So Itô's formula gives

$$M_t = M_0 + \int_0^t f_x(W_s, s)\, dW_s + \int_0^t \left(f_t + \frac{1}{2} f_{xx}\right)(W_s, s)\, ds \qquad \text{for } 0 \leq t < T.$$

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- Now one can check by laborious analysis that the function $f(x, t)$ satisfies the partial differential equation (PDE) $f_t + \frac{1}{2} f_{xx} = 0$; or one can use the fact that the canonical decomposition of a special semimartingale (like the martingale M) is unique.

Example IV (continued...)

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- Any of these leads to the conclusion that the ds-integral must vanish identically (because it is continuous and adapted, hence predictable, and of finite variation).
- By letting $t \nearrow T$, we therefore obtain the representation

$$h(W_T) = M_T = M_0 + \int_0^T f_x(W_s, s)\, dW_s$$

of the random variable $h(W_T)$ as an initial value M_0 plus a stochastic integral with respect to the Brownian motion W.
- A more general result in that direction will be given later.

- An *Itô process* is a stochastic process of the form

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s \qquad \text{for } t \geq 0$$

for some Brownian motion W, where μ and σ are predictable processes satisfying appropriate integrability conditions (e.g. $\int_0^T (|\mu_s| + |\sigma_s|^2) \, ds < \infty$ P-a.s. for every $T < \infty$).

Example V

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- More generally, X, μ, W could be vector-valued and σ matrix-valued.
- For any C^2-function f, the process $f(X)$ is then again an Itô process, and Itô's formula gives

$$f(X_t) = f(X_0) + \int_0^t \left(f'(X_s)\mu_s + \frac{1}{2} f''(X_s)\sigma_s^2 \right) ds + \int_0^t f'(X_s)\sigma_s \, dW_s.$$

This is another good *exercise* for using Itô's formula.

Example VI

▷ For any two real-valued semimartingales X and Y, the *product rule* is obtained by applying Itô's formula with the function $f(x, y) = xy$.

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- The result says that

$$X_t Y_t = X_0 Y_0 + \int_0^t Y_{s-} \, dX_s + \int_0^t X_{s-} \, dY_s + [X, Y]_t$$

or, in differential notation,

$$d(XY) = Y_- \, dX + X_- \, dY + d[X, Y].$$

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or, in differential notation,

$$d(XY) = Y_-\, dX + X_-\, dY + d[X, Y].$$

- If both X and Y are continuous, this yields

$$d(XY) = Y\, dX + X\, dY + d\langle X, Y\rangle.$$

- Let $W = (W_t)_{t \geq 0}$ be a Brownian motion, $a < 0 < b$ and

$$\tau_{a,b} := \inf \left\{ t \geq 0 \,\middle|\, W_t > b \text{ or } W_t < a \right\}$$

the first time that BM leaves the interval $[a, b]$ around 0.

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Then classical results about the *ruin problem for Brownian motion* say that

$$E[\tau_{a,b}] = |a|\, b \qquad (\text{so that } \tau_{a,b} < \infty \ P\text{-a.s.})$$

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$$P[W_{\tau_{a,b}} = b] = \frac{|a|}{b-a} = 1 - P[W_{\tau_{a,b}} = a].$$

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It is also known that $E[W_{\tau_{a,b}}] = 0$.

Example VII (continued...)

- In order to compute the *covariance* of $\tau_{a,b}$ and $W_{\tau_{a,b}}$, we start with the function $f(x,t) = -\frac{1}{3}x^3 + tx$. Then clearly $f_t + \frac{1}{2}f_{xx} \equiv 0$ so that Itô's formula shows that

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$$M_t := f(W_t, t) = 0 + \int_0^t f_x(W_s, s)\, dW_s.$$

Therefore M is, like W, a continuous local martingale.
But

$$M_t^{\tau_{a,b}} = M_{t\wedge\tau_{a,b}} = -\frac{1}{3}\left(W_t^{\tau_{a,b}}\right)^3 + (t \wedge \tau_{a,b})W_t^{\tau_{a,b}}$$

is bounded by a constant for $t \leq T$ as $|W^{\tau_{a,b}}| \leq \max(|a|, b)$, and so $M^{\tau_{a,b}}$ is a martingale on $[0, T]$ for each $T < \infty$.

- This gives

$$0 = E\left[M_0^{\tau_{a,b}}\right] = E\left[M_T^{\tau_{a,b}}\right] = -\frac{1}{3}E\left[W^3_{\tau_{a,b}\wedge T}\right] + E\left[(\tau_{a,b} \wedge T)W_{\tau_{a,b}\wedge T}\right],$$

and letting $T \to \infty$ yields by dominated convergence, also because $\tau_{a,b} \in L^1$, that

$$0 = -\frac{1}{3}E\left[W^3_{\tau_{a,b}}\right] + E\left[\tau_{a,b}W_{\tau_{a,b}}\right].$$

Example VII (continued...)

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- Hence we find

$$\mathrm{Cov}\left(\tau_{a,b}, W_{\tau_{a,b}}\right) = E\left[\tau_{a,b}W_{\tau_{a,b}}\right] = \frac{1}{3}E\left[W_{\tau_{a,b}}^3\right] = \frac{1}{3}|a|b(b-|a|),$$

where the last equality is obtained by computing with the known (two-point) distribution of $W_{\tau_{a,b}}$.

Girsanov's Theorem

Motivation:

- Previously, we have seen that the family of semimartingales is *invariant* under a transformation by a C^2-function, i.e. $f(X)$ is a semimartingale whenever X is a semimartingale and $f \in C^2$.
- In this section, our goal is to show that the class of semimartingales is also invariant under a change to an equivalent probability measure.

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- Previously, we have seen that the family of semimartingales is *invariant* under a transformation by a C^2-function, i.e. $f(X)$ is a semimartingale whenever X is a semimartingale and $f \in C^2$.
- In this section, our goal is to show that the class of semimartingales is also invariant under a change to an equivalent probability measure.

Setting:

- ▷ Suppose we have P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. We fix $T \in (0, \infty)$ and assume only that $Q \approx P$ on $\mathcal{F}_T$.
- ▷ If we have this for every $T < \infty$, we call Q and P *locally equivalent* and write $Q \overset{\text{loc}}{\approx} P$.
- ▷ For an infinite horizon, this is usually strictly weaker than $Q \approx P$.

To start, let us for simplicity fix $T \in (0, \infty)$ and suppose that $Q \approx P$ on $\mathcal{F}_T$. Denote by

$$Z_t := E_P\left[\frac{dQ|_{\mathcal{F}_T}}{dP|_{\mathcal{F}_T}} \,\middle|\, \mathcal{F}_t\right] \qquad \text{for } 0 \leq t \leq T$$

the *density process* of Q with respect to P on $[0, T]$, choosing an RCLL version of this P-martingale on $[0, T]$.

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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the *density process* of Q with respect to P on $[0, T]$, choosing an RCLL version of this P-martingale on $[0, T]$.

Because $Q \approx P$ on $\mathcal{F}_T$, we have $Z > 0$ on $[0, T]$, and because Z is a P-(super)martingale, this implies that also $Z_- > 0$ on $[0, T]$ by the so-called minimum principle for supermartingales.

Lemma. *Suppose that $Q \approx P$ on $\mathcal{F}_T$. Then:*

1) *For $s \leq t \leq T$ and every U_t which is $\mathcal{F}_t$-measurable and either ≥ 0 or in $L^1(Q)$, we have the Bayes formula*

$$E_Q[U_t \,|\, \mathcal{F}_s] = \frac{1}{Z_s} E_P[Z_t U_t \,|\, \mathcal{F}_s] \qquad Q\text{-a.s.}$$

Lemma. *Suppose that $Q \approx P$ on $\mathcal{F}_T$. Then:*

1) *For $s \leq t \leq T$ and every U_t which is $\mathcal{F}_t$-measurable and either ≥ 0 or in $L^1(Q)$, we have the Bayes formula*

$$E_Q[U_t \,|\, \mathcal{F}_s] = \frac{1}{Z_s} E_P[Z_t U_t \,|\, \mathcal{F}_s] \qquad Q\text{-a.s.}$$

2) *An adapted process $Y = (Y_t)_{0 \leq t \leq T}$ is a (local) Q-martingale if and only if the product ZY is a (local) P-martingale.*

The next result now proves the announced fundamental result.

Theorem. (Girsanov) *Suppose that* $Q \overset{\text{loc}}{\approx} P$ *with density process* Z*. If* M *is a local* P*-martingale null at 0, then*

$$\widetilde{M} := M - \int \frac{1}{Z}\, d[Z, M]$$

is a local Q*-martingale null at 0. In particular, every* P*-semimartingale is also a* Q*-semimartingale (and vice versa, by symmetry).*

Proof. The second assertion is very easy to prove; we simply write

$$X = X_0 + M + A = X_0 + \widetilde{M} + \left(A + \int \frac{1}{Z} \, d[Z, M] \right) = X_0 + \widetilde{M} + \widetilde{A}$$

and observe that $\widetilde{A} := A + \int \frac{1}{Z} \, d[Z, M]$ is of finite variation.

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For proving the first assertion, we first note that by the product rule,

$$ZM - [Z, M] = \int Z_- \, dM + \int M_- \, dZ$$

is a local P-martingale, like M and Z, so that by the previous lemma

$$Y := M - \frac{1}{Z}[Z, M] \text{ is a local } Q\text{-martingale.}$$

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is a local P-martingale, like M and Z, so that by the previous lemma

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Again using the product rule we get

$$\frac{1}{Z}[Z, M] = \int [Z, M]_- \, d\left(\frac{1}{Z}\right) + \int \frac{1}{Z_-} \, d[Z, M] + \left[\frac{1}{Z}, [Z, M]\right]. \qquad (1)$$

Because $[Z, M]$ is of finite variation, the last term equals

$$\left[\frac{1}{Z}, [Z, M]\right] = \sum \Delta\left(\frac{1}{Z}\right) \Delta[Z, M] = \int \Delta\left(\frac{1}{Z}\right) d[Z, M]$$

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Because $\frac{1}{Z}$ is a local Q-martingale, so is the stochastic integral $V := \int [Z, M]_- \, d\left(\frac{1}{Z}\right)$, because its integrand is locally bounded.

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Putting things together we have

$$\widetilde{M} = M - \int \frac{1}{Z}\, d[Z, M] = Y + V,$$

so that $\widetilde{M}$ is also a local Q-martingale. **q.e.d.**

Girsanov Theorem II

- Because $Z_- > 0$ (on $[0, T]$ or $[0, \infty)$, respectively), we can define a local P-martingale null at 0 by $L := \int \frac{1}{Z_-} \, dZ$.

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  $$Z = Z_0\, \mathcal{E}(L) > 0 \qquad \text{with a local } P\text{-martingale } L \text{ null at } 0.$$

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$$Z = Z_0 \, \mathcal{E}(L) > 0 \qquad \text{with a local } P\text{-martingale } L \text{ null at } 0.$$

  It is also clear that L is continuous if and only if Z is continuous.

Theorem. (Girsanov, continuous version) *Suppose that $Q \overset{\text{loc}}{\approx} P$ with a density process Z which is continuous. Write $Z = Z_0 \, \mathcal{E}(L)$. If M is a local P-martingale null at 0, then*

$$\widetilde{M} := M - [L, M] = M - \langle L, M \rangle$$

is a local Q-martingale null at 0. Moreover, if W is a P-Brownian motion, then $\widetilde{W}$ is a Q-Brownian motion. In particular, if $L = \int \nu \, dW$ for some $\nu \in L^2_{\text{loc}}(W)$, then $\widetilde{W} = W - \langle \int \nu \, dW, W \rangle = W - \int \nu_s \, ds$ so that the P-Brownian motion $W = \widetilde{W} + \int \nu_s \, ds$ becomes under Q a Brownian motion with (instantaneous) drift ν.

Proof. Because $Z = Z_0\,\mathcal{E}(L)$ satisfies $dZ = Z_-\,dL$, we have $[Z, M] = \int Z_-\,d[L, M]$ and hence $\int \frac{1}{Z}\,d[Z, M] = \int \frac{Z_-}{Z}\,d[L, M] = [L, M]$ by continuity of Z.

Girsanov Theorem II - Proof

Proof. Because $Z = Z_0\,\mathcal{E}(L)$ satisfies $dZ = Z_-\,dL$, we have $[Z, M] = \int Z_-\,d[L, M]$ and hence $\int \frac{1}{Z}\,d[Z, M] = \int \frac{Z_-}{Z}\,d[L, M] = [L, M]$ by continuity of Z.

So the first assertion follows directly from the last theorem, and $[L, M] = \langle L, M \rangle$ because L is continuous like Z.

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The assertion for $\widetilde{W}$ needs more work as it relies on the so-called Lévy characterisation of Brownian motion that we have not discussed here.
q.e.d.

- In all the above discussions, we have assumed that Q is already given, and we have then studied its effect on given processes. But in mathematical finance, we often want to proceed the other way round: We start with a process $S = (S_t)_{0 \leq t \leq T}$ of discounted asset prices and then want to find or construct some $Q \approx P$ on $\mathcal{F}_T$ such that S becomes a local Q-martingale.

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- We now discuss how to tackle this problem by reverse-engineering the preceding theory.

- We start with a local P-martingale L null at 0 and define $Z := \mathcal{E}(L)$ so that Z is like L a local P-martingale, with $Z_0 = 1$. If we also have $\Delta L > -1$ (and this happens of course in particular if L is continuous), then we have in addition $Z > 0$.

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- ▷ Suppose now that Z is a true P-martingale on $[0, T]$. This amounts to imposing suitable extra conditions on L.
- ▷ Then we can define a probability measure $Q \approx P$ on $\mathcal{F}_T$ by setting $dQ := Z_T \, dP$, and the density process of Q with respect to P on $[0, T]$ is then the P-martingale Z, by construction.

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- ▷ Suppose now that Z is a true P-martingale on $[0, T]$. This amounts to imposing suitable extra conditions on L.
- ▷ Then we can define a probability measure $Q \approx P$ on $\mathcal{F}_T$ by setting $dQ := Z_T \, dP$, and the density process of Q with respect to P on $[0, T]$ is then the P-martingale Z, by construction.
- ▷ If L is continuous, then also Z is continuous.

- Now start with an $\mathbb{R}^d$-valued process $S = (S_t)_{0 \le t \le T}$ and suppose that S is a P-semimartingale. For each i, the coordinate S^i can then (in general non-uniquely) be written as

$$S^i = S^i_0 + M^i + A^i$$

with a local P-martingale M^i and an adapted process A^i of finite variation, both null at 0.

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with a local P-martingale M^i and an adapted process A^i of finite variation, both null at 0.
Then

$$\widetilde{M}^i = M^i - \int \frac{1}{Z}\, d[Z, M^i]$$

is a local Q-martingale, and of course we have

$$S^i = S_0^i + \widetilde{M}^i + \left(A^i + \int \frac{1}{Z}\, d[Z, M^i]\right) = S_0^i + \widetilde{M}^i + \widetilde{A}^i.$$

- So S^i is a local Q-martingale (or, equivalently, Q is an ELMM for S^i) if and only if

$$\widetilde{A}^i = A^i + \int \frac{1}{Z} \, d[Z, M^i] \quad \text{is a local } Q\text{-martingale.}$$

Girsanov - Comments

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$$\widetilde{A}^i = A^i + \int \frac{1}{Z}\, d[Z, M^i] \quad \text{is a local } Q\text{-martingale.}$$

- One sufficient condition is of course that

$$A^i + \int \frac{1}{Z}\, d[Z, M^i] \equiv 0.$$

- So S^i is a local Q-martingale (or, equivalently, Q is an ELMM for S^i) if and only if
$$\widetilde{A}^i = A^i + \int \frac{1}{Z}\, d[Z, M^i] \quad \text{is a local } Q\text{-martingale.}$$
- One sufficient condition is of course that
$$A^i + \int \frac{1}{Z}\, d[Z, M^i] \equiv 0.$$
- This should be viewed as a condition on Z or, equivalently, on L. In general, because $dZ = Z_- \, dL$, we have
$$[Z, M^i] = \int Z_- \, d[L, M^i]$$
and $\Delta Z = Z_- \Delta L$, hence
$$Z = Z_- + \Delta Z = Z_-(1 + \Delta L)$$
and so
$$Z_-/Z = \frac{1}{1 + \Delta L}.$$

- So in terms of L, the condition can be written as

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- If L is continuous, this simplifies further to

$$A^i + \langle L, M^i \rangle \equiv 0;$$

this could alternatively also be derived directly from the last theorem.

Remark

The latter results can also be derived more directly:

- Suppose again that $Z = \mathcal{E}(L)$ is a true P-martingale > 0 on $[0, T]$, and define $Q \approx P$ on $\mathcal{F}_T$ by $dQ := Z_T \, dP$.

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- Suppose again that $Z = \mathcal{E}(L)$ is a true P-martingale > 0 on $[0, T]$, and define $Q \approx P$ on $\mathcal{F}_T$ by $dQ := Z_T \, dP$.
- Then, S is a local Q-martingale if and only if ZS is a local P-martingale, and so we compute, using the product rule and $dZ = Z_- \, dL$,

$$\begin{aligned} d(ZS^i) &= S^i_- \, dZ + Z_- \, dS^i + d[Z, S^i] \\ &= S^i_- \, dZ + Z_- \, dM^i + Z_-(dA^i + d[L, S^i]). \end{aligned}$$

Remark

▷ Because both Z and M^i, and hence also their stochastic integrals above, are local P-martingales, we see that Q is an ELMM for S^i if and only if $A^i + [L, S^i]$ is a local P-martingale. A sufficient condition for this is that

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If L is continuous, this again simplifies to

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because then $[L, A^i] = \sum \Delta L \, \Delta A^i \equiv 0$.

Itô's representation theorem

Motivation:

- Our goal in this section is to describe all martingales with respect to a filtration $\mathbb{F}$ which is *generated by a Brownian motion* W.
- This deep structural result goes back to Kiyosi Itô and is the mathematical explanation for the completeness of the Black–Scholes model that we shall see in the next chapter.

Motivation:

- Our goal in this section is to describe all martingales with respect to a filtration $\mathbb{F}$ which is *generated by a Brownian motion* W.
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Setup:

▷ We start with a Brownian motion $W = (W_t)_{t \geq 0}$ in $\mathbb{R}^m$ defined on a probability space $(\Omega, \mathcal{F}, P)$ without an a priori filtration. We define

$$\begin{aligned} \mathcal{F}_t^0 &:= \sigma(W_s, s \leq t) \qquad \text{for } t \geq 0, \\ \mathcal{F}_\infty^0 &:= \sigma(W_s, s \geq 0), \end{aligned}$$

and construct the filtration $\mathbb{F}^W = (\mathcal{F}_t^W)_{0 \leq t \leq \infty}$ by adding to each $\mathcal{F}_t^0$ all subsets of P-nullsets in $\mathcal{F}_\infty^0$ to obtain $\mathcal{F}_t^W = \mathcal{F}_t^0 \vee \mathcal{N}$.

- This so-called *P-augmented filtration* $\mathbb{F}^W$ is then P-complete (in $(\Omega, \mathcal{F}^0_\infty, P)$, to be precise) by construction, and one can show, by using the strong Markov property of Brownian motion, that $\mathbb{F}^W$ is also automatically right-continuous (so that it satisfies the usual conditions).

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- We usually call $\mathbb{F}^W$, somewhat misleadingly, the filtration generated by W. One can show that W is also a Brownian motion with respect to $\mathbb{F}^W$. The key point is to argue that $W_t - W_s$ is still independent of $\mathcal{F}^W_s \supseteq \mathcal{F}^0_s$, even though $\mathcal{F}^W_s$ contains some sets from $\mathcal{F}^0_\infty$.

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- If one works on $[0, T]$, one replaces ∞ by T. Then $\mathcal{F}^0_\infty$ is not needed because we use the P-nullsets from the "last" σ-field $\mathcal{F}^0_T$.

Theorem. (Itô's representation theorem) *Let* $W = (W_t)_{t \geq 0}$ *be a Brownian motion in* $\mathbb{R}^m$. *Then every random variable* $H \in L^1(\mathcal{F}_\infty^W, P)$ *has a unique representation as*

$$H = E[H] + \int_0^\infty \psi_s \, dW_s \qquad P\text{-a.s.}$$

for an $\mathbb{R}^m$*-valued integrand* $\psi \in L^2_{\mathrm{loc}}(W)$ *with the additional property that* $\int \psi \, dW$ *is a* $(P, \mathbb{F}^W)$*-martingale on* $[0, \infty]$ *(and therefore uniformly integrable).*

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Remark. The assumptions on H say that H is integrable and $\mathcal{F}^W_\infty$-measurable. The latter means intuitively that $H(\omega)$ can depend in a measurable way on the entire trajectory $W_\cdot(\omega)$ of Brownian motion, but not on any other source of randomness.

Corollary. *Suppose the filtration $\mathbb{F} = \mathbb{F}^W$ is generated by a Brownian motion W in $\mathbb{R}^m$. Then:*

1) *Every (real-valued) local $(P, \mathbb{F}^W)$-martingale L is of the form $L = L_0 + \int \gamma \, dW$ for some $\mathbb{R}^m$-valued process $\gamma \in L^2_{\mathrm{loc}}(W)$.*
2) *Every local $(P, \mathbb{F}^W)$-martingale is continuous.*

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Proof. For a localizing sequence $(\tau_k)_{k \in \mathbb{N}}$, each $(L - L_0)^{\tau_k}$ is a uniformly integrable martingale N^k, say, and therefore of the form

$$N^k_t = E\left[N^k_\infty \,\middle|\, \mathcal{F}^W_t\right] \qquad \text{for } 0 \le t \le \infty,$$

for some $N^k_\infty \in L^1(\mathcal{F}^W_\infty, P)$. So the IRT theorem and the martingale property of $\int \psi^k \, dW$ give that $N^k = \int \psi^k \, dW$ (note that $N^k_0 = 0$). In particular, $N^k = (L - L_0)^{\tau_k}$ is continuous. As $\tau_k \nearrow \infty$, L is continuous, and γ is obtained by piecing together the ψ^k. **q.e.d.**

While the above results are remarkable, the next result is bizarre. Note that in its formulation, the filtration $\mathbb{F}$ is even allowed to be general; but of course we could also take $\mathbb{F} = \mathbb{F}^W$.

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Theorem. (Dudley) *Let $W = (W_t)_{t \geq 0}$ be a Brownian motion with respect to P and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. As usual, set*

$$\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t = \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t \right).$$

Then every $\mathcal{F}_\infty$-measurable random variable H with $|H| < \infty$ P-a.s. (for example every $H \in L^1(\mathcal{F}_\infty, P)$) can be written as

$$H = \int_0^\infty \psi_s \, dW_s \qquad P\text{-a.s.}$$

for some integrand $\psi \in L^2_{\mathrm{loc}}(W)$.

Dudley - Comments

- Note that there is no constant in the representation of H in Dudley's result. Note also that we could for instance take for H a constant and represent this as a stochastic integral of Brownian motion. This kind of implies that the integrand ψ cannot be nice.

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In fact:

1) The stochastic integral process $\int \psi \, dW$ is of course a *local martingale, but* in general *not a martingale* on $[0, \infty]$. If it were, it would have constant expectation 0, which would imply that $E[H] = 0$.

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1) The stochastic integral process $\int \psi \, dW$ is of course a *local martingale, but* in general *not a martingale* on $[0, \infty]$. If it were, it would have constant expectation 0, which would imply that $E[H] = 0$.
2) The representation by ψ is *not unique*. In fact, one can easily construct some bounded predictable $\bar{\psi}$ with $0 < \int_0^\infty \bar{\psi}_s^2 \, ds < \infty$ P-a.s. (so that $\bar{\psi} \not\equiv 0$), but nevertheless $\int_0^\infty \bar{\psi}_s \, dW_s = 0$ P-a.s. Of course, ψ and $\psi + \bar{\psi}$ then represent the same H, but they are different in a nontrivial way.

3) In terms of finance, the integrands ψ are not nice at all:
   - ▷ The representation $1 = \int_0^\infty \psi_s \, dW_s$ looks suspiciously like creating the riskless payoff 1 out of zero initial capital with a self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \psi)$, which would be arbitrage. (But of course, that φ is not admissible, see below.)

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   - Note that $\int \psi \, dW$ cannot be bounded from below in general, because if it were, then $\int \psi \, dW$ would be a local martingale uniformly bounded from below, hence a supermartingale, and this implies that we must have $E[H] \leq 0$.

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   - Note that $\int \psi \, dW$ cannot be bounded from below in general, because if it were, then $\int \psi \, dW$ would be a local martingale uniformly bounded from below, hence a supermartingale, and this implies that we must have $E[H] \leq 0$.

4) It is not important for the above results that we work on the infinite interval $[0, \infty)$ or $[0, \infty]$; everything could be done equally well on $[0, T]$ for any $T \in (0, \infty)$. **This does not only hold for Dudley's result...**