Mathematical Foundations for Finance

Chapter VII: The Black-Scholes Formula

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Table of contents

The Black-Scholes Model

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Generalities

The *Black–Scholes model* or *Samuelson model* is the continuous-time analogue of the Cox–Ross–Rubinstein binomial model we have seen at length in earlier chapters.

Like the latter, it is too simple to be realistic, but still very popular because it allows many explicit calculations and results.

It also serves as a basic starting point or reference model.

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To set up the model, we start with a fixed time horizon $T \in (0, \infty)$ and a probability space $(\Omega, \mathcal{F}, P)$ on which there is a Brownian motion $W = (W_t)_{0 \leq t \leq T}$.

We take as *filtration* $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ the one generated by W and augmented by the P-nullsets from $\mathcal{F}_T^0 := \sigma(W_s; s \leq T)$ so that $\mathbb{F} = \mathbb{F}^W$ satisfies the usual conditions under P.
(We shall see soon that this choice of filtration is important.)

The *financial market model* has two basic traded assets: a bank account with constant continuously compounded *interest rate* $r \in \mathbb{R}$, and a *risky asset* (usually called stock) having two parameters $\mu \in \mathbb{R}$ and $\sigma > 0$.

Undiscounted prices are given by

$$\widetilde{S}_t^0 = e^{rt}, \tag{1}$$

$$\widetilde{S}_t^1 = S_0^1 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right) \tag{2}$$

with a constant $S_0^1 > 0$. Applying Itô's formula yields

$$d\widetilde{S}_t^0 = \widetilde{S}_t^0 r \, dt, \tag{3}$$

$$d\widetilde{S}_t^1 = \widetilde{S}_t^1 \mu \, dt + \widetilde{S}_t^1 \sigma \, dW_t. \tag{4}$$

This can be rewritten as

$$\frac{d\widetilde{S}_t^0}{\widetilde{S}_t^0} = r\,dt, \tag{5}$$

$$\frac{d\widetilde{S}_t^1}{\widetilde{S}_t^1} = \mu\,dt + \sigma\,dW_t. \tag{6}$$

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- ▷ In the same way, the relative price change of the stock has a part $\mu\,dt$ giving a growth at rate μ, and a second part $\sigma\,dW_t$ "with mean 0 and variance $\sigma^2\,dt$" that causes random fluctuations.
  We call μ the *drift* (rate) and σ the (instantaneous) *volatility* of $\widetilde{S}^1$.

As usual, we pass to quantities *discounted* with $\widetilde{S}^0$. So we have $S^0 = \widetilde{S}^0/\widetilde{S}^0 \equiv 1$, and $S^1 = \widetilde{S}^1/\widetilde{S}^0$ is by (1) and (2) given by

$$S_t^1 = S_0^1 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right)t\right). \tag{7}$$

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$$S^1_t = S^1_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right)t\right). \tag{7}$$

Either from (7) or from (3), (4), we obtain via Itô's formula that S^1 solves the SDE

$$dS^1_t = S^1_t\big((\mu - r)\,dt + \sigma\,dW_t\big). \tag{8}$$

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$$dS_t^1 = S_t^1\big((\mu - r)\,dt + \sigma\,dW_t\big). \tag{8}$$

For later use, we observe that this gives

$$d\langle S^1\rangle_t = (S_t^1)^2\sigma^2\,dt \tag{9}$$

for the *quadratic variation* of S^1, because $\langle W\rangle_t = t$.

Remark. Because the coefficients μ, r, σ are all constant, the undiscounted prices $(\widetilde{S}^0, \widetilde{S}^1)$, the discounted prices (S^0, S^1), the discounted stock price S^1 alone, and the Brownian motion W all generate the same filtration.

This means that there is no compromise between mathematical convenience (the filtration $\mathbb{F}$ is generated by W) and financial modelling (the filtration is generated by information about prices).

As in discrete time, we should like to have an *equivalent martingale measure* for the discounted stock price process S^1. To get an idea how to find this, we rewrite (8) as

$$dS_t^1 = S_t^1 \sigma \left(dW_t + \frac{\mu - r}{\sigma} dt \right) = S_t^1 \sigma \, dW_t^*, \tag{10}$$

with $W^* = (W_t^*)_{0 \le t \le T}$ defined by

$$W_t^* := W_t + \frac{\mu - r}{\sigma} t = W_t + \int_0^t \lambda \, ds \qquad \text{for } 0 \le t \le T.$$

The quantity

$$\lambda := \frac{\mu - r}{\sigma}$$

is often called the instantaneous *market price of risk* or infinitesimal *Sharpe ratio* of S^1.

By looking at Girsanov's theorem, we see that W^* is a Brownian motion under the probability measure Q^* given by

$$\frac{dQ^*}{dP} := \mathcal{E}\left(-\int \lambda\, dW\right)_T = \exp\left(-\lambda W_T - \frac{1}{2}\lambda^2 T\right),$$

whose density process with respect to P is

$$Z_t^* = \mathcal{E}\left(-\int \lambda\, dW\right)_t = \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right) \qquad \text{for } 0 \leq t \leq T.$$

By (10), the stochastic integral process

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from (10) by Itô's formula.

All in all, then, S^1 admits an equivalent martingale measure, explicitly given by Q^*, and so we expect that S^1 should be "arbitrage-free" in any reasonable sense. However, we cannot make this precise here before defining more carefully what "trading strategy", "self-financing", "arbitrage opportunity" etc. should mean in this context.

Remark. Suppose Q is any probability measure equivalent to P on $\mathcal{F}_T$ and denote its P-density process by $Z = (Z_t)_{0 \leq t \leq T}$.

Then we can write $Z = Z_0 \, \mathcal{E}(L)$, where L is a local $(P, \mathbb{F})$-martingale null at 0.

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Then we can write $Z = Z_0 \, \mathcal{E}(L)$, where L is a local $(P, \mathbb{F})$-martingale null at 0.

Since $\mathbb{F}$ is generated by W, Itô's representation theorem says that

$$L = \int \nu_s \, dW_s \qquad \text{for some } \nu \in L^2_{\mathrm{loc}}(W)$$

and therefore $dZ_t = Z_{t-} \, dL_t = Z_t \nu_t \, dW_t$ (as Z is automatically continuous like L).

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The product rule and the rules for computing covariations of stochastic integrals give

$$\begin{aligned} d(Z_t S_t^1) &= Z_t\, dS_t^1 + S_t^1\, dZ_t + d\langle Z, S^1\rangle_t \\ &= Z_t S_t^1(\mu - r)\, dt + Z_t S_t^1 \sigma\, dW_t + S_t^1 Z_t \nu_t\, dW_t + Z_t \nu_t S_t^1 \sigma\, d\langle W, W\rangle_t \\ &= Z_t S_t^1(\sigma + \nu_t)\, dW_t + Z_t S_t^1 \sigma(\lambda + \nu_t)\, dt, \end{aligned}$$

using that $\mu - r = \sigma\lambda$.

On the left-hand side, we have by assumption a local P-martingale, and on the right-hand side, the dW-integral is also a local P-martingale.
Therefore the last term,

$$A_t := \int_0^t Z_s S_s^1 \sigma(\lambda + \nu_s)\, ds \qquad \text{for } 0 \leq t \leq T,$$

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This implies that $\nu_s \equiv -\lambda$, because Z, S^1, σ are all > 0, and so we get

$$Z = Z_0\, \mathcal{E}(L) = Z_0\, \mathcal{E}\left(\int \nu\, dW \right) = Z_0\, \mathcal{E}\left(- \int \lambda\, dW \right).$$

Finally, Z_0 has P-expectation 1 and is measurable with respect to $\mathcal{F}_0 = \mathcal{F}_0^W$ which is P-trivial (because W_0 is constant P-a.s.). So $Z_0 = 1$ and therefore

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So we expect that the Black–Scholes model is not only "arbitrage-free", but also "complete" in a suitable sense.

Note that the latter point (as well as the above proof of uniqueness) depends via Itô's representation theorem in a crucial way on the assumption that the filtration $\mathbb{F}$ is generated by W. $\diamond$

Now take any $H \in L^0_+(\mathcal{F}_T)$ and view H as a random *payoff* (in discounted units) due at time T.

Recall that $\mathbb{F}$ is generated by W and that $W^*_t = W_t + \lambda t$, $0 \leq t \leq T$, is a Q^*-Brownian motion.

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Because λ is deterministic, W and W^* generate the same filtration, and so we can also apply Itô's representation theorem with Q^* and W^* instead of P and W.

So if H is also in $L^1(Q^*)$, the Q^*-martingale $V^*_t := E_{Q^*}[H \,|\, \mathcal{F}_t]$, $0 \le t \le T$, can be represented as

$$V^*_t = E_{Q^*}[H] + \int_0^t \psi^H_s \, dW^*_s \qquad \text{for} \quad 0 \le t \le T,$$

with some unique $\psi^H \in L^2_{\mathrm{loc}}(W^*)$ such that $\int \psi^H \, dW^*$ is a Q^*-martingale.

Recall from (10) that

$$dS_t^1 = S_t^1 \sigma \, dW_t^* .$$

So if we define for $0 \leq t \leq T$

$$\vartheta_t^H := \frac{\psi_t^H}{S_t^1 \sigma}, \qquad \eta_t^H := V_t^* - \vartheta_t^H S_t^1$$

(which are both predictable because ψ^H is), then we can interpret $\varphi^H = (\vartheta^H, \eta^H)$ as a *trading strategy* whose discounted value process is given by

$$V_t(\varphi^H) = \vartheta_t^H S_t^1 + \eta_t^H S_t^0 = V_t^* \qquad \text{for } \ 0 \leq t \leq T,$$

and which is *self-financing* in the (usual) sense that for $0 \leq t \leq T$

$$V_t(\varphi^H) = V_t^* = V_0^* + \int_0^t \psi_u^H \, dW_u^* = V_0(\varphi^H) + \int_0^t \vartheta_u^H \, dS_u^1 . \tag{12}$$

Moreover,

$$V_T(\varphi^H) = V_T^* = H \qquad \text{a.s.}$$

shows that the strategy φ^H *replicates* H, and

$$\int \vartheta^H \, dS^1 = V(\varphi^H) - V_0(\varphi^H) = V^* - E_{Q^*}[H] \geq -E_{Q^*}[H]$$

(because $V^* \geq 0$, as $H \geq 0$) shows that ϑ^H is admissible (for S^1) in the usual sense.

In summary, then, every $H \in L^1_+(\mathcal{F}_T, Q^*)$ is *attainable* in the sense that it can be replicated by a dynamic strategy trading in the stock and the bank account in such a way that the strategy is self-financing and admissible, and its value process is a Q^*-martingale.

In that sense, we can say that the Black–Scholes model is complete.

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By the same arguments as in discrete time, we then also obtain the arbitrage-free value at time t of any payoff $H \in L^1_+(\mathcal{F}_T, Q^*)$, say V^H_t, as its conditional expectation

$$V^H_t = V_t(\varphi^H) = V^*_t = E_{Q^*}[H \mid \mathcal{F}_t]$$

under the unique equivalent martingale measure Q^* for S^1.

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2) Itô's representation theorem gives the *existence* of a strategy, but does not tell us how it looks. To get more explicit results, additional structure (for the payoff H) and more work is needed [$\rightarrow$ *exercise*].
3) The SDE (8) for discounted prices is

$$\frac{dS_t^1}{S_t^1} = (\mu - r)\, dt + \sigma\, dW_t,$$

and this is rather restrictive as μ, r, σ are all constant. An obvious *extension* is to allow the coefficients μ, r, σ to be (suitably integrable) predictable processes, or possibly functionals of S or $\widetilde{S}$.

This brings up several issues:

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b) If μ, r, σ are stochastic processes that depend on extra randomness apart from W, we have to work in a larger filtration and a result like Itô's representation theorem is perhaps no longer available.
Typical examples are *stochastic volatility* models where σ usually depends on a second Brownian motion as well, or *credit risk* models where the default of an asset often involves the jump of some process.

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Typical examples are *stochastic volatility* models where σ usually depends on a second Brownian motion as well, or *credit risk* models where the default of an asset often involves the jump of some process.

c) Even if μ, r, σ are predictable with respect to the filtration $\mathbb{F}$ generated by W, the process $W^* = W + \int \lambda_s \, ds$ in general does not generate $\mathbb{F}$, but only a smaller filtration.
Fortunately, there is still a representation result with respect to W^* and Q^*, but one must work a little to prove this.

Remarks

4) From the point of view of finance, the *natural filtration* to work with would be the one generated by S or $\widetilde{S}$, i.e. by prices, not by W.
   From the explicit formulae (1), (2), one can see that $\widetilde{S}$ and W generate the same filtrations when the coefficients μ, r, σ are deterministic (as already been pointed out in this section).
   But in general (i.e. for more general coefficients), working with the price filtration is rather difficult because this is hard to describe.

◇

Markovian Payoffs and PDEs

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The presentation in the previous section is often called the *martingale approach* to valuing options, for obvious reasons.

If one has more structure for the payoff H (and, in more general models, also for S), an alternative method involves the use of partial differential equations (PDEs) and is thus called the *PDE approach*.

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If one has more structure for the payoff H (and, in more general models, also for S), an alternative method involves the use of partial differential equations (PDEs) and is thus called the *PDE approach*.

Suppose that the (discounted) payoff is of the form $H = h(S^1_T)$ for some measurable function $h \geq 0$ on $\mathbb{R}_+$. We also suppose that H is in $L^1(Q^*)$. One example discussed in detail in the next section is the European call option on $\widetilde{S}^1$ with maturity T and undiscounted strike $\widetilde{K}$:

$$H = (\widetilde{S}^1_T - \widetilde{K})^+ / \widetilde{S}^0_T = (S^1_T - \widetilde{K}e^{-rT})^+.$$

Our goal, for general h, is to compute the value process V^* and the strategy ϑ^H more explicitly.

We start with the *value process*.

Because we have $V_t^* = E_{Q^*}[H \,|\, \mathcal{F}_t] = E_{Q^*}[h(S_T^1) \,|\, \mathcal{F}_t]$, we look at (11) and write

$$S_T^1 = S_t^1 \frac{S_T^1}{S_t^1} = S_t^1 \exp\left(\sigma(W_T^* - W_t^*) - \frac{1}{2}\sigma^2(T-t)\right).$$

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In the last term, the first factor S_t^1 is obviously $\mathcal{F}_t$-measurable.

Moreover, W^* is a Q^*-Brownian motion with respect to $\mathbb{F}$, and so in the second factor, $W_T^* - W_t^*$ is under Q^* independent of $\mathcal{F}_t$ and has an $\mathcal{N}(0, T-t)$-distribution.

Therefore we get

$$V_t^* = E_{Q^*}[h(S_T^1) \mid \mathcal{F}_t] = v(t, S_t^1) \tag{13}$$

with the function $v(t, x)$ given by

$$\begin{aligned} v(t,x) &= E_{Q^*}\left[h\left(x\exp\left(\sigma(W_T^* - W_t^*) - \frac{1}{2}\sigma^2(T-t)\right)\right)\right] \qquad (14) \\ &= E_{Q^*}\left[h\left(xe^{\sigma\sqrt{T-t}\,Y - \frac{1}{2}\sigma^2(T-t)}\right)\right] \\ &= \int_{-\infty}^{\infty} h\left(xe^{\sigma\sqrt{T-t}\,y - \frac{1}{2}\sigma^2(T-t)}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,dy, \end{aligned}$$

where $Y \sim \mathcal{N}(0,1)$ under Q^*.

This already gives a fairly precise structural description of V_t^* as a function of (t and) S_t^1, instead of a general $\mathcal{F}_t$-measurable random variable.

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This gives, writing subscripts in the function v for partial derivatives,

$$\begin{aligned} dV_t^* &= dv(t, S_t^1) \qquad (15) \\ &= v_t(t, S_t^1)\, dt + v_x(t, S_t^1)\, dS_t^1 + \frac{1}{2} v_{xx}(t, S_t^1)\, d\langle S^1 \rangle_t \\ &= v_x(t, S_t^1) \sigma S_t^1\, dW_t^* + \left(v_t(t, S_t^1) + \frac{1}{2} v_{xx}(t, S_t^1) \sigma^2 (S_t^1)^2 \right) dt \end{aligned}$$

by using (10) and (9).

Note that V^* is a local (even a true) Q^*-martingale, by its definition, and so is the dW^*-term on the right-hand side above.

Therefore the dt-term on the right-hand side of (15) is at the same time continuous and adapted and of finite variation, and a local Q^*-martingale. Hence it must vanish, and so (15) and (12) yield

$$v_x(t, S_t^1)\, dS_t^1 = dV_t^* = \vartheta_t^H\, dS_t^1$$

so that we obtain the *strategy* explicitly as

$$\vartheta_t^H = \frac{\partial v}{\partial x}(t, S_t^1), \tag{16}$$

i.e. as the spatial derivative of v, evaluated along the trajectories of S^1.

A closer look at the above argument allows us to extract some information about the function v as well.

In fact, the vanishing of the dt-term means that the function $v_t(t,x) + \frac{1}{2}v_{xx}(t,x)\sigma^2 x^2$ must vanish along the trajectories of the space-time process $(t, S^1_t)_{0<t<T}$.

But each S^1_t is by (11) lognormally distributed and hence has all of $(0,\infty)$ in its support. So the support of the space-time process contains $(0,T)\times(0,\infty)$, and so $v(t,x)$ must satisfy the (linear, second-order) *PDE*

$$0 = \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} \qquad \text{on } (0,T)\times(0,\infty). \tag{17}$$

Moreover, the definition of v via (13) gives the *boundary condition*

$$v(T,\cdot\,) = h(\,\cdot\,) \qquad \text{on } (0,\infty), \tag{18}$$

because $v(T, S^1_T) = V^*_T = H = h(S^1_T)$ and the support of the distribution of S^1_T contains $(0,\infty)$.

Moreover, the definition of v via (13) gives the *boundary condition*

$$v(T, \cdot\,) = h(\,\cdot\,) \qquad \text{on } (0, \infty), \tag{18}$$

because $v(T, S_T^1) = V_T^* = H = h(S_T^1)$ and the support of the distribution of S_T^1 contains $(0, \infty)$.

So if we cannot compute the integral in (14) explicitly, we can at least obtain $v(t, x)$ *numerically* by solving the PDE (17), (18).

Remarks

Instead of using the above probabilistic argument, one can also derive the PDE (17) *analytically*.

Instead of using the above probabilistic argument, one can also derive the PDE (17) *analytically*.

Using in (14) the substitution

$$u = x \exp(\sigma \sqrt{T-t}\, y - \frac{1}{2}\sigma^2 (T-t))$$

gives

$$y = \frac{\log \frac{u}{x} + \frac{1}{2}\sigma^2 (T-t)}{\sigma \sqrt{T-t}},$$

and hence

$$dy = \frac{1}{u\sigma\sqrt{T-t}}\, du\,.$$

So we can write

$$v(t,x) = \int_0^\infty h(u) \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \frac{1}{u} \exp\left(-\frac{(\log \frac{u}{x} + \frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)} \right) du \, .$$

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One can check that v can be differentiated by differentiating under the integral sign, and by *brute force computations*, one can then check in that way that v indeed satisfies (17).

(The deeper reason behind this is that the density function $\varphi(t,z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}$ of an $\mathcal{N}(0,t)$-distribution satisfies the heat equation $\varphi_t = \frac{1}{2}\varphi_{zz}$.)

Undiscounted Version of the PDE

When comparing the PDE (17), (18) to some of those found in the literature, one might be puzzled by the simple form of (17).

This is because we have expressed everything in discounted units.

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If the *undiscounted* payoff is $\widetilde{H} = \tilde{h}(\widetilde{S}_T^1)$ and the undiscounted value at time t is $\tilde{v}(t, \widetilde{S}_t^1)$, we have the relations

$$\tilde{h}\big(\widetilde{S}_T^1\big) = \tilde{h}\big(e^{rT} S_T^1\big) = \widetilde{H} = e^{rT} H = e^{rT} h(S_T^1)$$

and

$$\tilde{v}\big(t, \widetilde{S}_t^1\big) = e^{rt} v(t, S_t^1)$$

so that

$$v(t, x) = e^{-rt} \tilde{v}(t, xe^{rt})$$

or

$$\tilde{v}(t, \tilde{x}) = e^{rt} v(t, \tilde{x}e^{-rt}).$$

For the function $\tilde{v}$, we can then compute the partial derivatives

$$\begin{aligned}\frac{\partial \tilde{v}}{\partial t}(t,\tilde{x}) &= r\tilde{v}(t,\tilde{x}) + e^{rt}\frac{\partial v}{\partial t}(t,\tilde{x}e^{-rt}) - e^{rt}\frac{\partial v}{\partial x}(t,\tilde{x}e^{-rt})\tilde{x}re^{-rt},\\ \frac{\partial \tilde{v}}{\partial \tilde{x}} &= e^{rt}\frac{\partial v}{\partial x}(t,\tilde{x}e^{-rt})e^{-rt} = \frac{\partial v}{\partial x}(t,\tilde{x}e^{-rt}),\\ \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} &= \frac{\partial^2 v}{\partial 2}(t,\tilde{x}e^{-rt})e^{-rt},\end{aligned}$$

and by plugging in, we obtain from (17) the PDE

$$0 = \frac{\partial \tilde{v}}{\partial t} + r\tilde{x}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \frac{1}{2}\sigma^2\tilde{x}^2\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} - r\tilde{v}$$

with the boundary condition

$$\tilde{v}(T,\cdot) = \tilde{h}(\cdot).$$

The Black-Scholes Formula

In the special case of a *European call option*, the value process and the corresponding strategy can be computed explicitly, and this has found widespread use in industry:

Consider the undiscounted payoff

$$\widetilde{H} = (\widetilde{S}_T^1 - \widetilde{K})^+.$$

Then $H = \widetilde{H}/\widetilde{S}_T^0 = (S_T^1 - \widetilde{K}e^{-rT})^+ =: (S_T^1 - K)^+$, and we obtain from (14) that the discounted value of H at time t is

$$V_t^* = E_{Q^*}\left[\left(xe^{\sigma\sqrt{T-t}\,Y - \frac{1}{2}\sigma^2(T-t)} - K\right)^+\right]\Bigg|_{x=S_t^1}.$$

Because we have $Y \sim \mathcal{N}(0,1)$ under Q^*, an elementary computation yields for $x > 0$, $a > 0$ and $b \geq 0$ that

$$E_{Q^*}\left[\left(xe^{aY-\frac{1}{2}a^2} - b\right)^+\right] = x\Phi\left(\frac{\log\frac{x}{b} + \frac{1}{2}a^2}{a}\right) - b\Phi\left(\frac{\log\frac{x}{b} - \frac{1}{2}a^2}{a}\right),$$

where

$$\Phi(y) = Q^*[Y \leq y] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

is the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$.

Plugging in $x = S_t^1$, $a = \sigma\sqrt{T-t}$, $b = K$ and then passing to undiscounted quantities yields the famous *Black–Scholes formula*:

$$\tilde{V}_t^{\widetilde{H}} = \tilde{v}\big(t, \widetilde{S}_t^1\big) = \widetilde{S}_t^1 \Phi(d_1) - \widetilde{K} e^{-r(T-t)} \Phi(d_2) \tag{19}$$

with

$$d_{1,2} = \frac{\log(\widetilde{S}_t^1/\widetilde{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{20}$$

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with

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Note that the drift μ of the stock does not appear here. This is analogous to the result that the probability p of an up move in the CRR binomial model does not appear in the binomial option pricing formula.

To compute the *replicating strategy*, we recall from (16) that the stock price holdings at time t are given by

$$\vartheta_t^H = \frac{\partial v}{\partial x}(t, S_t^1).$$

Moreover, $v(t,x) = e^{-rt}\tilde{v}(t, xe^{rt})$ so that

$$\frac{\partial v}{\partial x}(t,x) = e^{-rt}\frac{\partial \tilde{v}}{\partial \tilde{x}}(t, xe^{rt})e^{rt} = \frac{\partial \tilde{v}}{\partial \tilde{x}}(t, xe^{rt}).$$

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Computing the above derivative explicitly [$\rightarrow$ *exercise*] gives

$$\vartheta_t^H = \frac{\partial \tilde{v}}{\partial \tilde{x}}(t, \widetilde{S}_t^1) = \Phi(d_1) = \Phi\left(\frac{\log(\widetilde{S}_t^1/\widetilde{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right), \quad (21)$$

which always lies between 0 and 1.

One very useful feature of the above result is that the explicit formula (19), (20) allows to compute all partial derivatives of the option price with respect to the various parameters. Some examples are (see lecture notes for more):

- *Delta*: the partial derivative with respect to $\widetilde{S}^1_t$, also called *hedge ratio*.
- *Gamma*: the second partial derivative with respect to $\widetilde{S}^1_t$; it measures the reaction of Delta to a stock price change.
- *Rho*: the partial derivative with respect to the interest rate r.
- *Vega*: the partial derivative with respect to the volatility σ.
- *Theta*: the partial derivative with respect to $T - t$, the time to maturity.

Of course, the above definitions per se make sense for any model. But in the Black–Scholes model, one has even explicit expressions for them.