Summary

Step 0 : Lebesgue Stieltjes integral for deterministic processes.

$\mu = \mu_+ - \mu_-$ where $\mu_{+,-}$ are ≥ 0 measures on $[0,T]$

càdlàg bounded variation $a : [0,T] \to \mathbb{R}$ $\iff$ $\mu : [0,T] \to \mathbb{R}$ signed measure

$a(t) = \mu([0,t])$.

$\to$ can define $\int_s^t h(x)\, da(x) = \int_{(s,t]} h(x)\, \mu(dx)$

Prop : $a : [0,T] \to \mathbb{R}$ càdlàg BV , $h \in L^1([0,T], |da|)$ $\Rightarrow$ $h \cdot a : [0,T] \to \mathbb{R}$ is càdlàg & BV

Example : $a : [0,1] \to \mathbb{R}$ $\quad a(t) = \begin{cases} 1 & t < \frac{1}{2} \\ 0 & t \geq \frac{1}{2} \end{cases}$

$V_a(0,1) = 1 \qquad \mu = \delta_0 - \delta_{1/2}$

Step 1 Lebesgue Stieltjes integral wrt càdlàg finite variation processes.

Def : A càdlàg adapted process X is a map $X : \Omega \times [0,\infty) \to \mathbb{R}$ st

i) X is càdlàg , i.e $X(\omega, \cdot) : [0,\infty) \to \mathbb{R}$ is càdlàg for all $\omega \in \Omega$

ii) X is adapted, i.e $X_t = X(\cdot, t) : \Omega \to \mathbb{R}$ is $\mathcal{F}_t$ measurable for all t.

Def : A càdlàg process A is a finite variation process if $A(\omega, \cdot) : [0,\infty) \to \mathbb{R}$ has finite variation for all $\omega \in \Omega$.

The total variation process V associated to a finite variation process A is

$V_t = \int_0^t |dA_s|$ $\to$ random variable defined ω-wise as in step 0 using ~~the~~ for each ω the signed measure μ_ω corresponding to $A_\cdot(\omega) = A(\omega, \cdot)$

Def : Let A be a càdlàg finite variation process and H be a process st

$$\forall \omega \in \Omega, \ \forall t \geq 0 : \int_0^t |H_s(\omega)|\, |dA_s(\omega)| < \infty$$

We define the stochastic integral of H wrt A by the process $((H \cdot A)_t)_{t \geq 0}$ given by

$$(H \cdot A)_t = \int_0^t H_s\, dA_s$$

$\to$ More precisely we define the stochastic integral ω-wise. Fix an $\omega \in \Omega$. $A(\omega, \cdot)$ is càdlàg finite variation process. By step 0, $\exists$ a corresponding signed measure μ_ω and we define

$$(H \cdot A)_t(\omega) = \int_0^t H_s(\omega)\, dA_s(\omega) \underset{\uparrow \text{ see step 0}}{=} \int_0^t H_s(\omega)\, \mu_\omega(ds)$$

Note $(H \cdot A) : \Omega \times [0,\infty) \to \mathbb{R}$ is a stochastic process!

We know from step 0 that this definition of $(H \cdot A)$ gives a càdlàg and finite variation process for each $\omega \in \Omega$. <u>Question</u> : is $(H \cdot A)$, when viewed as a function of ω adapted, <u>i.e</u> $X(\cdot, t) : \Omega \to \mathbb{R}$ $\in \mathcal{F}_t$ $\forall t$?

$\to$ need a condition on H : H must be predictable.

<u>Def</u> : The predictable (or previsible) σ-algebra $\mathcal{P}$ is the σ-algebra on $\Omega \times [0, \infty)$ generated by all adapted left-continuous processes.

$A \in \mathcal{P}$ is thus the pre-image of a Borel set $B \in \mathcal{B}(\mathbb{R})$ by an adapted left-continuous process $Y : \Omega \times [0, \infty) \to \mathbb{R}$:

$$\cancel{A \in} \mathcal{P} = \sigma\left(Y^{-1}(B) : B \in \mathcal{B}(\mathbb{R}) \right)$$

$$\cancel{= \sigma(\{\omega}$$

One can show that $\mathcal{P}$ can equivalently be defined as the σ-algebra on $\Omega \times [0, \infty)$ generated by sets of the form

$$E \times (s, t] \quad , \quad E \in \mathcal{F}_s \quad , \quad s < t$$

A process $H : \Omega \times [0, \infty) \to \mathbb{R}$ is said to be predictable if it is $\mathcal{P}$-measurable, <u>i.e</u> $H^{-1}(B) \in \mathcal{P}$ $\forall B \in \mathcal{B}(\mathbb{R})$.

Equivalently, H is predictable if it is the pointwise limit of bounded elementary processes (see below)

<u>Def</u> : A process $H : \Omega \times [0, \infty) \to \mathbb{R}$ is a bounded elementary (also called simple / simple predictable) process if

$$H(\omega, t) = \sum_{i=1}^{n} h_{i-1}(\omega) \, \mathbb{1}_{(t_{i-1}, t_i]}(t)$$

<u>Rmk</u> A notation does not correspond to the one in lecture notes (shift in indices)

for bounded (real-valued) $\mathcal{F}_{t_{i-1}}$-measurable random variables and $0 \leq t_0 < t_1 < \dots < t_n < \infty$.

<u>Note</u> By construction simple predictable processes are

- left-continuous for each $\omega \in \Omega$ (because $t \mapsto \mathbb{1}_{(t_{i-1}, t_i]}(t)$ is left-continuous)
- predictable (because adapted and left-continuous)

Facts giving sone intuitions about predictable processes:

i) Simple processes and their pointwise limits are predictable

ii) Adapted left-continuous processes are predictable

In particular adapted continuous processes (e.g Brownian Motion) are predictable.

iii) Let H be predictable. Then $H_t \in \mathcal{F}_{t-} := \sigma(\mathcal{F}_s, s<t)$

In particular this shows that adapted but not left-continuous processes might not be predictable

<u>Example</u>: A poisson process (N_t) is adapted (to it's natural filtration) but not predictable since $N_t \notin \mathcal{F}_{t-}$.

<u>Thm</u>: Let A be a càdlàg finite variation process, and let H be a predictable process. Then the ω-wise defined Lebesgue Stieltjes integral

$$(H\cdot A)_t(\omega) := \int_0^t H_s(\omega)\, dA_s(\omega) = \int_0^t H_s(\omega)\, \mu_\omega(ds)$$

defines an adapted finite variation process $H\cdot A : \Omega\times[0,\infty) \to \mathbb{R}$.

We would like to define the stochastic integral wrt local martingales (and later semimartingales). Unfortunately our above construction fails because one can show that if X is a continuous local martingale starting at 0 and having finite variation, then $X_t = 0 \ \forall t$ a.s.

Hence we cannot construct the stochastic integral pathwise (ω by ω) using the relation between càdlàg bounded variation processes and signed measures. We need to find another way....

~~<u>Idea</u>: define the stoch. integral wrt a local continuous martingales bounded in L^2 by first defining it for simple processes H.~~

~~For a general predictable H, we need to use results construct the stoch. integral as a limit process in a suitable L^2 sense.~~

~~Finally we extend our theory to integration wrt local mart. and semimartingales~~

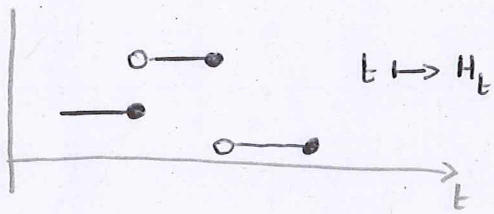
Step 2 ~~Step~~ : In order to define the stochastic integral $(H \cdot M)$ for a large class of integrands H and integrators M we proceed step by step.

Step A Define $(H \cdot M)$ for $H \in b\mathcal{E}$ a bounded elementary process arbitrarly M (in particular for square integrable martingale).

$$H = \sum_{i=0}^{n-1} h_i \cdot \mathbb{1}_{(t_i, t_{i+1}]} \qquad (\text{i.e } H_t(\omega) = \sum_{i=0}^{n-1} h_i(\omega) \cdot \mathbb{1}_{(t_i, t_{i+1}]}(t))$$

for some $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \dots < t_n$ and each h_i • bounded
• $\mathcal{F}_{t_i}$ - measurable.

Note : Such processes are left continuous
A typical path (for a fixed $\omega \in \Omega$) looks



For $H \in b\mathcal{E}$ and M arbitrary, we define the stochastic integral as

$$(H \cdot M)_t = \sum_{i=0}^{n-1} h_i \left(M_{t_{i+1} \wedge t} - M_{t_i \wedge t} \right)$$

i.e the stochastic integral $(H \cdot M) = ((H \cdot M)_t)_{t \geq 0}$ is a stochastic process and for a fixed time step t and a fixed $\omega \in \Omega$, the random variable $(H \cdot M)_t$ takes the value

$$(H \cdot M)_t(\omega) = \sum_{i=0}^{n-1} h_i(\omega) \cdot \left(M_{t_{i+1} \wedge t}(\omega) - M_{t_i \wedge t}(\omega) \right)$$

When defining the stoch. integral for $H \in b\mathcal{E}$ this way we note that an Isometry property holds. This Lemma, called Itô's isometry, is crucial in order to extend our definition to a larger class of integrands H.

<u>Lemma</u>

Suppose M is a square integrable martingale (i.e $M_t \in L^2$ for all $t \geq 0$). For every $H \in b\mathcal{E}$, the stochastic integral process $H \cdot M = \int H \, dM$ is then also a square integrable martingale, and we have the isometry property

$$\mathbb{E}\Big[\underbrace{(H \cdot M)^2_\infty}_{\substack{\text{stoch integral} \\ \text{as defined} \\ \text{in step A}}} \Big] = \mathbb{E}\Big[\Big(\int_0^\infty H_s \, dM_s \Big)^2 \Big]$$

$$= \mathbb{E}\Big[\underbrace{\int_0^\infty H_s^2 \, d[M]_s}_{\text{Lebesgue Stieltjes integral}} \Big]$$

since $t \mapsto [M]_t$ is increasing (hence of finite variation)

We can reinterpret this Lemma by introducing appropriate normed spaces.

Let $\mathcal{M}_0^2 = \{$ all càdlàg martingales $N = (N_t)_{t \geq 0}$ null at 0 which satisfy $\sup_{t \geq 0} \mathbb{E}[N_t^2] < \infty \}$

Moreover set $\|N\|_{\mathcal{M}_0^2} = \Big(\sup_{t \geq 0} \mathbb{E}[N_t^2] \Big)^{1/2}$ for $N \in \mathcal{M}_0^2$

<u>Exercise</u> prove that $\| \cdot \|_{\mathcal{M}_0^2}$ is a norm. (hint you will need Doob's L^2 inequality to prove $\|N\|_{\mathcal{M}_0^2} = 0 \Rightarrow N \equiv 0$)

Recall the following basic results from probability theory:

if $N \in \mathcal{M}_0^2$, then

- $N_t \underset{t \to \infty}{\longrightarrow} N_\infty$ a.s and in L^2 (see martingale convergence theorems)

- $N_\infty \in L^2$ by Fatou's lemma.

- $(N_t^2)_{t \geq 0}$ is a submartingale by Jensen's inequality so $t \mapsto \mathbb{E}[N_t^2]$ is increasing and in particular

$$\mathbb{E}[N_\infty^2] = \sup_{t \geq 0} \mathbb{E}[N_t^2] \quad \text{and so} \quad \|N\|_{\mathcal{M}_0^2} = (\mathbb{E}[N_\infty^2])^{1/2}$$

- Doob's L^2 inequality says

$$\mathbb{E}[\sup_{t \geq 0} N_t^2] \leq 4\, \mathbb{E}[N_\infty^2]$$

- the process $(N_t)_{0 \leq t \leq \infty}$ defined up to ∞ is still a martingale and Doob's maximal inequality implies that two martingales N and N' which have the same final value (i.e $N_\infty = N'_\infty$ P-as) must coincide P-as at every time step.

We can therefore identify $N \in \mathcal{M}_0^2$ with it's limit $N_\infty \in L^2(\mathcal{F}_\infty, P)$

This makes $(\mathcal{M}_0^2, \|\cdot\|_{\mathcal{M}_0^2})$ a Hilbert space with the norm

$$\|N\|_{\mathcal{M}_0^2} = (\mathbb{E}[N_\infty^2])^{1/2}$$

and the scalar product

$$(N, N')_{\mathcal{M}_0^2} = \mathbb{E}[N_\infty N'_\infty]$$

We also introduce the space

$$L^2(M) = L^2(\Omega \times (0,\infty), \mathcal{P}, P_M)$$

$$= \{ \text{all (equivalence classes of) predictable } H = (H_t)_{t\geq 0} \text{ such that } \|H\|_{L^2(M)} := \left(\mathbb{E}_M[H^2]\right)^{1/2} = \left(\mathbb{E}\left[\int_0^\infty H_s^2 \, d[M]_s\right]\right)^{1/2} < \infty \}$$

expectation with respect to the prob. measure P_M defined on $(\Omega \times (0,\infty), \mathcal{P})$

$\mathcal{P}$ ↳ predictable σ-field

by

$$E_M[Y] := \mathbb{E}\left[\int_0^\infty Y_s(\omega) \, d[M]_s(\omega)\right] \quad (*) \quad \text{for } Y \geq 0 \text{ predictable}$$

Note: $(*)$ indeed defines a probability measure on $\mathcal{P}$ because a set $A \in \mathcal{P}$ iff the function $\mathbb{1}_A$ is predictable and so one can define

$$P_M(A) := \mathbb{E}_M[\mathbb{1}_A] \quad \text{using } (*)$$

With these two spaces ($\mathcal{M}_0^2$ & $L^2(M)$) we can reformulate Itô's lemma as follows:

Suppose M is a square integrable martingale. Then for every $H \in b\mathcal{E}$, the mapping

$$\begin{aligned} b\mathcal{E} &\longrightarrow \mathcal{M}_0^2 \\ H &\longmapsto H \cdot M \end{aligned}$$

is linear and we have the isometry

$$\|(H\cdot M)\|^2_{\mathcal{M}_0^2} = \mathbb{E}\left[(H\cdot M)_\infty^2\right] = \mathbb{E}\left[\left(\int_0^\infty H_s \, dM_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^\infty H_s^2 \, d[M]_s\right] = \|H\|^2_{L^2(M)}$$

So Itô's isometry lemma shows that the stochastic integration viewed as a mapping

$$\begin{aligned} b\mathcal{E} &\longrightarrow \mathcal{M}_0^2 \\ H &\longmapsto H\cdot M \end{aligned}$$

is an isometry (distance preserving transformation) when we equip the spaces $b\mathcal{E}$ and $\mathcal{M}_0^2$ with the norms

- $$\|H\|^2_{L^2(M)} := \mathbb{E}\left[\int_0^\infty H_s^2 \, d[M]_s\right]$$

$$\|(H\cdot M)\|^2_{\mathcal{M}_0^2} = \mathbb{E}\left[\left(\int_0^\infty H_s \, dM_s\right)^2\right]$$

We can therefore extend our def. of stoch integral using the continuous linear extension theorem stated below:

<u>Thm</u> (continuous linear extension theorem)
Every bounded linear transformation $T: X \to Y$ from a normed vector space X to a complete normed vector space Y can be uniquely extended to a bounded linear transformation $\bar{T}: \bar{X} \to Y$ from the closure $\bar{X}$ of X to Y.

By Itô's lemma, the stoch int viewed as a mapping $b\mathcal{E} \to \mathcal{M}_0^2$ is linear and bounded from the normed vector space $(b\mathcal{E}, \|\cdot\|_{L^2(M)})$ to the complete normed vector space $(\mathcal{M}_0^2, \|\cdot\|_{\mathcal{M}_0^2})$

↳ completeness admitted.

So by the above theorem, H can be extended to the closure (with respect to $\|\cdot\|_{L^2(M)}$) of $b\mathcal{E}$. It turns out that when $M \in \mathcal{M}_0^2$, the closure of $b\mathcal{E}$ in the $\|\cdot\|_{L^2(M)}$ norm is $L^2(M)$. This leads us to step B.

$\underline{\text{Step B}}$ Extend the definition to $H \in L^2(M)$. This works well for $M \in \mathcal{M}_0^2$ because in that case one can show that $b\mathcal{E}$ is dense in $L^2(M)$ (i.e the closure of $b\mathcal{E}$ in $L^2(M)$ is $L^2(M)$).

By the previous arguments, we can extend (uniquely) the def of $(H \cdot M)$ to $H \in L^2(M)$ for $M \in \mathcal{M}_0^2$ and the properties of the map

$b\mathcal{E} \longrightarrow \mathcal{M}_0^2$ are preserved, i.e the resulting $(H \cdot M)$ is again
$H \longmapsto (H \cdot M)$ a martingale in $\mathcal{M}_0^2$ and still satisfies the isometry property.

The extension is done in the following way:

① The closure of $b\mathcal{E}$ in the $\|\cdot\|_{L^2(M)}$ norm is $L^2(M)$ so $b\mathcal{E}$ is dense in $L^2(M)$.

Take $H \in L^2(M)$ for a fixed $M \in \mathcal{M}_0^2$

By density of $b\mathcal{E}$ in $L^2(M)$, $\exists$ a sequence $H^n \in b\mathcal{E}$ st H^n converges to H in the $\|\cdot\|_{L^2(M)}$ norm, i.e

$$\lim_{n \to \infty} \| H^n - H \|_{L^2(M)} = 0$$

② By Itô's isometry property we have

$$\| H^n - H \|_{L^2(M)} = \| (H^n \cdot M) - (H \cdot M) \|_{\mathcal{M}_0^2}$$

and hence

$$\lim_{n \to \infty} \| (H^n \cdot M) - (H \cdot M) \|_{\mathcal{M}_0^2} = 0$$

i.e $(H^n \cdot M)$ converges to $(H \cdot M)$ in the $\|\cdot\|_{\mathcal{M}_0^2}$ norm

③ We can therefore define

$$(H\cdot M)_t^{(\omega)} := \left(\lim_{n\to\infty} (H^n \cdot M)\right)_t^{(\omega)}$$

where H^n is a sequence in $b\mathcal{E}$ converging to H in the $L^2(M)$ norm and the lim on the RHS is taken in the $\|\cdot\|_{\mathcal{M}_0^2}$-norm.

Problem : This works only if $M \in \mathcal{M}_0^2$. However that is a too strong assumption as for instance the Brownian motion is not in $\mathcal{M}_0^2$ as $\sup_{t\geq 0} \mathbb{E}[W_t^2] = \sup_{t\geq 0} t = \infty$.

This leads us to step C.

Step C Extend the definition to $H \in L^2_{loc}(M)$ and $M \in \mathcal{M}^2_{0,loc}$ via localisation

1) Define the space of locally square integrable local martingales $\mathcal{M}^2_{0,loc}$

A local martingale M null at 0 is called locally square integrable, and we write $M \in \mathcal{M}^2_{0,loc}$ if there exists a sequence of stopping times $\tau_n \nearrow \infty$ $\mathbb{P}$-as st the stopped process $M^{\tau_n} \in \mathcal{M}_0^2$ for all n

2) Define the space $L^2_{loc}(M)$

A predictable process $H \in L^2_{loc}(M)$ if there exists a sequence of stopping times $\tau_n \nearrow \infty$ $\mathbb{P}$-as st

$$H \cdot \mathbb{1}_{]\!]0,\tau_n]\!]} \in L^2(M) \qquad \text{for all } n$$

NB : $]\!]0,\tau_n]\!] = \{(\omega,t) \in \Omega \times (0,\infty) \mid 0 < t \leq \tau_n(\omega)\}$ denotes the so called stochastic interval.

<u>Easy example</u> Brownian Motion is in $\mathcal{M}^2_{0,loc}$ because for

$\tau_n = \inf\{t>0 \mid W_t > n\} \nearrow \infty$

↳ <u>exercise</u>: why?

we have $|W_{t\wedge\tau_n}| \leq n$ because $|W_{t\wedge\tau_n}| = \begin{cases} |W_t| < n & \text{if } t<\tau_n \\ |W_{\tau_n}| = n & \text{if } t\geq \tau_n \end{cases}$

uses the continuity of the Brownian paths

and so $\sup\limits_{t\geq 0} \mathbb{E}\left[|W_{t\wedge\tau_n}|^2\right] \leq n < \infty$ for each n.

Note that by definition of $\mathcal{M}^2_{0,loc}$ and $L^2_{loc}(M)$

$H \mathbb{1}_{]\!]0,\tau_n]\!]} \in L^2(M^{\tau_n})$

$M^{\tau_n} \in L^2(M)$

and so the stoch. integral $(H\cdot \mathbb{1}_{]\!]0,\tau_n]\!]} \bullet M^{\tau_n})$ is well defined and gives a local martingale in $\mathcal{M}^2_0(M^{\tau_n})$.

The idea is to define

$$(H\bullet M) := (H\cdot\mathbb{1}_{]\!]0,\tau_n]\!]} \bullet M^{\tau_n}) \quad \text{on }]\!]0,\tau_n]\!]$$

To make this rigourous, one has to show that

i) this gives a def on all $\Omega\times(0,\infty)$ $\longrightarrow$ ok because $\tau_n \nearrow \infty$
$\Rightarrow]\!]0,\tau_n]\!] \nearrow \Omega\times(0,\infty)$

ii) the def. is consistent, i.e that the def on $]\!]0,\tau_{n+1}]\!] \supseteq]\!]0,\tau_n]\!]$ does not clash with the def on $]\!]0,\tau_n]\!]$.

↳ omitted in this course.

To get an intuition about the space $L^2_{loc}(M)$, consider the special case where M is continuous local mart. null at 0 $(M \in \mathcal{M}^c_{0,loc})$.

Then $M \in \mathcal{M}^2_{0,loc}$ because it is even locally bounded. Indeed using the localising sequence

$$\tau_n = \inf\{t > 0 \mid |M_t| > n\}$$

we get $|M^{\tau_n}_t| = |M_{t \wedge \tau_n}| \leq n$ for each n.

↳ exercise : explain why (hint same argument as for Brownian Motion on previous page).

It turns out that when M is continous (or more generally if M has integrable jumps) we can characterise $L^2_{loc}(M)$ as

$$L^2_{loc}(M) = \{ \text{all predictable process } H = (H_t)_{t \geq 0} \text{ such that } \int_0^t H_s^2 \, d[M]_s < \infty \ P\text{-as for each } t \geq 0 \}$$

Remark

1) The main difference between $L^2(M)$ and $L^2_{loc}(M)$ in this case is that we only require

$$\int_0^t H_s^2 \, d[M]_s < \infty \quad \forall t \geq 0$$

instead of $\mathbb{E}\left[\int_0^t H_s^2 \, d[M]_s \right] < \infty \quad \forall t \geq 0$

Note that this is much weaker requirement as

$\mathbb{E} Y < \infty \Rightarrow Y < \infty$ but the converse is false in general

↳ exercise: find an example where $Y < \infty$ but $\mathbb{E} Y = \infty$

To see why we can "ommit" the expectation, we just come back to the definition of $L^2_{loc}(M)$ and use the particular localising sequence

$$\tau_n = \inf\{t > 0 : \int_0^t H_s^2 \, d[M]_s = n\}.$$

it can be shown that $(\tau_n)_{n \geq 0}$ is a localising sequence and clearly

$$\mathbb{E}\Big[\underbrace{\int_0^\infty H_s^2 \, \mathbb{1}_{]\!]0,\tau_n]\!]} \, d[M]_s}_{\leq n}\Big] \leq n < \infty$$

so $H \cdot \mathbb{1}_{]\!]0,\tau_n]\!]} \in \mathcal{L}^2(M)$ and hence $H \in L^2_{loc}(M)$

(for details see exercise sheet).

2) When M is continuous, the optional quadratic variation $[M]$ and the sharp bracket processes $\langle M \rangle$ coincide.

Indeed M continuous $\Rightarrow \Delta M = 0$

$\Rightarrow (\Delta M)^2 = 0 = \Delta[M] \Rightarrow [M]$ continuous

But $[M]$ adapted by def so in this case also predictable (because adapted continuous processes are predictable

$\Rightarrow [M] = \langle M \rangle$ by uniqueness of $\langle M \rangle$.

~~This concludes step 2. The next steps will be done next week by Prof Parkes:~~

~~Step D~~

⊕ See Remarks p 85-86 ⎫
⊕ See Properties p 88-90 ⎭ done on slides.

This concludes step C. Next week you will further extend the stochastic integration ~~as summarised below~~ to semimartingales:

Step D Define $(H \cdot X)$ for H predictable and locally bounded

X semimartingale