Mathematical Foundations for Finance

Exercise sheet 1

*This sheet contains material which is **fundamental** for this course and **assumed to be known**. An exception to this are questions 4 and 5 which are technically more involved as well as the coding question 6 which is not part of the examinable material. We consider questions 4, 5 and 6 as BONUS problems. Results and facts from measure-theoretic probability theory will be often used in the lectures and exercise classes. To get an idea of the prerequesites, read the Appendix of your lecture notes (pages 129 to 136). The material **should be familiar** and in case it is not, you are expected to catch up as quicly as possible. The book Probability Essentials by Jean Jacod and Philip Protter contains all results needed and can be downloaded for free from Springer (within the ETH network or using VPN). A possible alternative to the above textbook are the ETH lecture notes for the standard course on Probability Theory, accessible from the MFF course website.*
Please upload your solutions until Wednesday, 29/09/2021, 12:00 using the link on the course website.

Exercise 1.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with $\Omega := \{UU, UD, DD, DU\}$, $\mathcal{F} := 2^\Omega$ and P defined by $P[\omega] := 1/4$ for all $\omega \in \Omega$. Let $Y_1, Y_2 : \Omega \to \mathbb{R}$ be two random variables with $Y_1(UU) = Y_1(UD) := 2$, $Y_1(DD) = Y_1(DU) := 1/2$, $Y_2(UU) = Y_2(DU) := 2$ and $Y_2(DD) = Y_2(UD) := 1/2$. Define the process $X = (X_k)_{k=0,1,2}$ by

$$X_0(\omega) = 8 \quad \text{for all } \omega \in \Omega,$$
$$X_k(\omega) = X_0(\omega) \prod_{i=1}^{k} Y_i(\omega) \quad \text{for } k = 1, 2.$$

(a) Write down explicitly the sequences of σ-fields $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ and $\mathbb{G} = (\mathcal{G}_k)_{k=0,1,2}$ defined by $\mathcal{F}_k := \sigma(X_i, 0 \le i \le k)$ and $\mathcal{G}_k := \sigma(X_k)$, $k = 0, 1, 2$.

(b) Show that $Z : \Omega \to \mathbb{R}$ defined by $Z(\omega) := 2X_1(\omega) + 1$ is $\mathcal{G}_1$-measurable.

(c) Are $\mathbb{F}$ and $\mathbb{G}$ filtrations on $(\Omega, \mathcal{F})$? Why or why not?

(d) Is X adapted to $\mathbb{F}$ or $\mathbb{G}$ (in case any of the former is a filtration on $(\Omega, \mathcal{F})$)?

(e) Try to give financial interpretations for X and $\mathbb{F}$.

Exercise 1.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable with $X \ge 0$ P-a.s. Prove that $E[X] = 0$ implies that $X = 0$ P-a.s.
Hint: Find a way to use the monotone convergence theorem.

Exercise 1.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space, X an integrable random variable and $\mathcal{G} \subseteq \mathcal{F}$ a σ-field. Then the P-a.s. unique random variable Z such that

- Z is $\mathcal{G}$-measurable and integrable,
- $E[X\mathbb{1}_A] = E[Z\mathbb{1}_A]$ for all $A \in \mathcal{G}$,

is called *conditional expectation of X given $\mathcal{G}$* and is denoted by $E[X \mid \mathcal{G}]$. (This is the formal definition of conditional expectation of X given $\mathcal{G}$; see Section 8.2 in the lecture notes.)

(a) Use the definition above to show that if X is $\mathcal{G}$-measurable, then $E[X \mid \mathcal{G}] = X$ P-a.s.

(b) Use the definition of conditional expectation to show that $E[E[X \mid \mathcal{G}]] = E[X]$.

(c) Use the definition of conditional expectation to show that if $P[A] \in \{0,1\}$ for all $A \in \mathcal{G}$, i.e. if $\mathcal{G}$ is P-trivial, then $E[X \mid \mathcal{G}] = E[X]$ P-a.s.

(d) Consider another integrable random variable Y on $(\Omega, \mathcal{F}, P)$, and two constants $a, b \in \mathbb{R}$. Use the definition of conditional expectation to show that $E[aX + bY \mid \mathcal{G}] = aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}]$ P-a.s.

(e) Suppose that $\mathcal{G}$ is generated by a finite partition of Ω, i.e. there exists a collection $(A_i)_{i=1}^n$ of sets $A_i \in \mathcal{F}$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mathcal{G} = \sigma(A_1, \ldots, A_n)$. Additionally, assume that $P[A_i] > 0$ for all $i = 1, \ldots, n$. Use the definition of conditional expectation to show that

$$E[X \mid \mathcal{G}] = \sum_{i=1}^n E[X \mid A_i] \mathbb{1}_{A_i} \qquad P\text{-a.s.}$$

Hint 1: Recall that $E[X \mid A_i] = E[X\mathbb{1}_{A_i}]/P[A_i]$ and try to write X as a sum of random variables each of which only takes non-zero values on a single A_i.
Hint 2: Check that any set $A \in \mathcal{G}$ is of the form $\cup_{j \in J} A_j$ for some $J \subseteq \{1, \ldots, n\}$.

Exercise 1.4 Let (U, V) be a two dimensional random vector. Suppose that the distribution of (U, V) admits a density function $f_{U,V}$ with respect to the Lebesgue measure. Let

$$f_U(u) = \int_{\mathbb{R}} f_{U,V}(u, v) dv$$

denote the marginal distribution of U and define the conditional density of V given U by

$$f_{V|U}(v|u) := \frac{f_{U,V}(u, v)}{f_U(u)}.$$

Consider a Borel measurable function $h : \mathbb{R} \to \mathbb{R}$ such that $h(V) \in L^1$ and define

$$g(u) := \int h(v) f_{V|U}(v|u) dv.$$

Show that $E[h(V) \mid U] = g(U)$.

Exercise 1.5 *The goal of this exercise is to show existence and uniqueness of the conditional expectation under the restrictive assumption of square integrability.*

Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-sigma-algebra of $\mathcal{F}$. Show that X admits a $\mathbb{P}$-almost surely unique conditional expectation, i.e that there exits a $\mathbb{P}$-almost surely unique random variable Y satisfying the following properties:

- Y is $\mathcal{G}$-measurable and integrable,
- $E[X\mathbb{1}_A] = E[Y\mathbb{1}_A]$ for all $A \in \mathcal{G}$.

The exercise involves Hilbert spaces and in particular an important result concerning orthogonal projections on convex closed subsets of a Hilbert space. For the sake of completeness we recall the definition of a Hilbert space and state this important result that you will need to use in question b).

A Hilbert space is a vector space H with an inner product $\langle f, g \rangle$ such that the norm defined by

$$||f|| := \sqrt{\langle f, f \rangle}$$

turns H into a complete metric space. In particular it can be shown that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Hilbert space with the inner product defined by

$$\langle X, Y\rangle := E\,[XY]\,.$$

Note that this integral is well defined by the Cauchy-Schwartz inequality and indeed defines an inner product on $L^2(\Omega, \mathcal{F}, \mathbb{P})$ (good bonus exercise). Equipped with the norm induced by this inner product, it is not too difficult to show that L^2 is complete and hence a Hilbert space. One of the nicest properties of Hilbert spaces is that one can generalize the notion of orthogonal projections from finite dimensional euclidiean vector spaces to (potentially) infinite dimensional Hilbert spaces as suggested by the following theorem: Let Γ be a closed convex subset of a Hilbert space H. Then for any point $x \in H$, there exists a unique point $\pi_\Gamma(x) \in \Gamma$ such that

$$||x - \pi_\Gamma(x)|| = \inf_{g\in\Gamma} ||x - g||.$$

Moreover $x - \pi_\Gamma(x) \in \Gamma^\perp$, where $\Gamma^\perp$ denotes the orthogonal complement of Γ, i.e.

$$\Gamma^\perp = \{x \in H s.t. \langle x, g\rangle = 0 \quad \forall g \in \Gamma\}.$$

(a) Show that if two random variables Y_1 and Y_2 satisfy the above properties, then we have $\mathbb{P}(Y_1 = Y_2) = 1$. This shows the $\mathbb{P}$-almost uniqueness.

Remains to show the existence. To do this we will apply the orthogonal projection theorem on the closed convex subset $L^2(\Omega, \mathcal{G}, \mathbb{P})$ of the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

(b) Show that $L^2(\Omega, \mathcal{G}, \mathbb{P})$ is convex in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

(c) Show that $L^2(\Omega, \mathcal{G}, \mathbb{P})$ is closed in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

(d) Deduce the existence of the conditional expectation of X given $\mathcal{G}$.

(e) Give a geometric interpretation of the above construction.

Exercise 1.6 *This is a **bonus** question on the Capital Asset Pricing Model (CAPM) and linear regression. This question is **not** part of the examinable material; it's purpose is just to motivate you to improve your programming skills. Programming skills are becoming more and more important especially for those of you wanting to work in the industry. The goal of the exercise is to compute the beta of a stock using linear regression.* The CAPM is a model that prices securities by examining the relationship between expected returns and risk. More precisely the model states that the return of a risky asset is given by

$$E[R_i] = r + \beta_i(E[R_m] - r) \tag{1}$$

where $E[R_i]$ and $E[R_m]$ are the expected returns of asset i and of the market respectively and r is the risk-free rate of interest. One can show that β_i has a closed from solution given by

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)}$$

The Beta of an asset is thus a measure of the sensitivity of its returns relative to a market benchmark (usually a market index). We could in principle calculate the Beta of a given asset using the above formula, however in practice it is easier to think of the Beta as the slope of the linear regression (1) and estimate it using Ordinary Least Squares method.

In this exercise we will estimate the Beta of Facebook relative to the S&P500 market index using historical data from 02/11/2014 to 19/11/2017. The data can be downloaded from the course web page ('FB.csv' and 'SP500.csv'). Your task is to complete **either** the below Python code **or** the R code in order to perform an Ordinary Least Squares Regression with Statsmodel. If your code works well, you should get a Beta value close to 0.58 which is the Beta value of Facebook quoted on Yahoo Finance on the 19/11/2017.

```
###### Python Code ########
###########################

# import libraries
import pandas as pd
import statsmodels.api as sm

'''
Download monthly prices of Facebook and S&P 500 index from 2014 to 2017
CSV file downloaded from Yahoo File
start period: 02/11/2014
end period: 30/11/2017
period format: DD/MM/YEAR
'''

# Step 1: Use pandas read_csv method to load the two csv files downloaded
    from the course web page

fb = # TO DO #
sp_500 = # TO DO #

# joining the closing prices of the two datasets
monthly_prices = pd.concat([fb['Close'], sp_500['Close']], axis=1)
monthly_prices.columns = ['FB', 'SP500']

# check the head of the dataframe
print(monthly_prices.head())

# calculate monthly returns
monthly_returns = # TO DO #
clean_monthly_returns = monthly_returns.dropna(axis=0)

# split dependent and independent variable
X = # TO DO #
y = # TO DO #

# Add a constant to the independent value
X1 = sm.add_constant(X)

# make regression model
model = # TO DO #

# fit model and print results
results = # TO DO #
print(results.summary())
```

```
###### R Code ########
#####################

# CAPM linear regression
# read in the csv files
fb <- # TO DO #
sp_500 <- # TO DO #

# joining the closing prices of the two datasets
```

```r
monthly_prices = cbind(fb['Close'], sp_500['Close'])
colnames(monthly_prices) = c('FB', 'SP500')

# check the head of the dataframe
head(monthly_prices)

# calculate monthly returns
# Denote n the number of time periods:
n <- nrow(monthly_prices)
monthly_returns <- ((monthly_prices[2:n, ] - monthly_prices[1:(n-1), ])/
    monthly_prices[1:(n-1), ])
# note that this is already the clean_monthly_returns from the Python
    code

# split dependent and independent variable
X = # TO DO #
y = # TO DO #

# the lm function automatically ads the intercept term
fit <- # TO DO #
summary(fit) # for interpretation see below
# Interpretation:
#   FB_returns = 0.020270 + 0.575091*SP500_returns

# visualize the results
coeff <- coefficients(fit)
plot(X,y, xlab = 'SP500 monthly returns', ylab = 'Facebook monthly
    returns')
abline(a=coeff[1], b=coeff[2], col=2, lwd=2)
```

Make sure you understand the summary of the linear regression and that in particular you can find the corresponding Beta value.