Mathematical Foundations for Finance

Exercise sheet 10

Please upload your solutions until Wednesday, 01/12/2021, 12:00 using the link on the course website.

Exercise 10.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions. Define

$$\tau_a := \inf\{t \geq 0 \mid W_t > a\}$$

for some $a > 0$.

(a) Prove that τ_a is a stopping time for all $a > 0$, and that we have $\tau_{a_1} \leq \tau_{a_2}$ P-a.s. for $a_1 < a_2$.
Hint 1: Use that fact that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then if $f(x) > a$ for some $x, a \in \mathbb{R}$, there exists a $y \in \mathbb{Q}$ arbitrarily close to x such that $f(y) > a$.
Hint 2: Use that the filtration is right-continuous, i.e. if $A \in \mathcal{F}_{t+1/n}$ for all $n \in \mathbb{N}$, then $A \in \mathcal{F}_t$.

(b) Prove that $P[\tau_a < \infty] = 1$ for all $a > 0$.
Hint: Use the global law of the iterated logarithm from Proposition V.1.2 in the lecture notes.

(c) Show that $W_{\tau_a} = a$ P-a.s. for all $a > 0$ and conclude that

$$E\left[W_{\tau_{a_2}} \,\middle|\, \mathcal{F}_{\tau_{a_1}}\right] \neq W_{\tau_{a_1}} \quad P\text{-a.s.},$$

for $a_1 < a_2$, proving that the stopping theorem (Theorem IV.2.1 in the lecture notes) fails for $\tau = \tau_{a_2}$ and $\sigma = \tau_{a_1}$.

(d) Prove that $\rho_a := \sup\{t \geq 0 | W_t > a\}$ is a stopping time. What values does it take?
Hint: Use that the filtration is P-complete, i.e. if $P[A] = 0$ for some $A \in \mathcal{F}$, then $A \in \mathcal{F}_0$.

Exercise 10.2 Let M be an RCLL local martingale null at 0 which satisfies $\sup_{0\leq s\leq T} |M_t| \in L^2$ for some $T \in \mathbb{R}$.

(a) Show that M is a square-integrable martingale on $[0, T]$.
Hint: Dominated convergence theorem.

(b) Let $[M]$ be the square bracket process of M. Show that

$$E\left[[M]_t - [M]_s \,|\, \mathcal{F}_s\right] = \mathrm{Var}[M_t - M_s \,|\, \mathcal{F}_s]$$

for all $0 \leq s \leq t \leq T$.

Exercise 10.3 On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, consider an adapted stochastic process $X = (X_t)_{t\geq 0}$ null at 0. Assume that it is integrable and has independent stationary increments, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same distribution as X_{t-s} for all $t > s$. (In particular, this is satisfied for any *Lévy process* $L = (L_t)_{t\geq 0}$ with $E[|L_1|] < \infty$).

(a) What conditions must $\left(E[X_t]\right)_{t\geq 0}$ satisfy in order to make X a $(P, \mathbb{F})$-supermartingale, a $(P, \mathbb{F})$-submartingale, or a $(P, \mathbb{F})$-martingale?

(b) Assume from now on that X is a square-integrable $(P, \mathbb{F})$-martingale. Prove that we have for all $t, s > 0$ that

$$E\left[X_t^2\right] + E\left[X_s^2\right] = E\left[X_{t+s}^2\right]$$

and deduce that $\left(E\left[X_t^2\right]\right)_{t \geq 0}$ is an increasing process.

(c) Use (b) to prove that $E\left[X_t^2\right] = tE\left[X_1^2\right]$ for all $t \geq 0$.
Hint: Prove the result first for $t = 1/n$ for all $n \in \mathbb{N}$. Deduce that it holds true for all $t \in \mathbb{Q}_+$ and use monotonicity to conclude.

(d) Prove that $\langle X \rangle_t = tE\left[X_1^2\right]$, for all $t \geq 0$.
Hint: Use your result from (c).