

Mathematical Foundations for Finance

Exercise sheet 10

Please upload your solutions until Wednesday, 01/12/2021, 12:00 using the link on the course website.

Exercise 10.1 Let $W = (W_t)_{t \geq 0}$ be a Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying the usual conditions. Define

$$\tau_a := \inf\{t \geq 0 \mid W_t > a\}$$

for some $a > 0$.

- (a) Prove that τ_a is a stopping time for all $a > 0$, and that we have $\tau_{a_1} \leq \tau_{a_2}$ P -a.s. for $a_1 < a_2$.
Hint 1: Use that fact that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then if $f(x) > a$ for some $x, a \in \mathbb{R}$, there exists a $y \in \mathbb{Q}$ arbitrarily close to x such that $f(y) > a$.
Hint 2: Use that the filtration is right-continuous, i.e. if $A \in \mathcal{F}_{t+1/n}$ for all $n \in \mathbb{N}$, then $A \in \mathcal{F}_t$.
- (b) Prove that $P[\tau_a < \infty] = 1$ for all $a > 0$.
Hint: Use the global law of the iterated logarithm from Proposition V.1.2 in the lecture notes.
- (c) Show that $W_{\tau_a} = a$ P -a.s. for all $a > 0$ and conclude that

$$E[W_{\tau_{a_2}} \mid \mathcal{F}_{\tau_{a_1}}] \neq W_{\tau_{a_1}} \quad P\text{-a.s.},$$

for $a_1 < a_2$, proving that the stopping theorem (Theorem IV.2.1 in the lecture notes) fails for $\tau = \tau_{a_2}$ and $\sigma = \tau_{a_1}$.

- (d) Prove that $\rho_a := \sup\{t \geq 0 \mid W_t > a\}$ is a stopping time. What values does it take?
Hint: Use that the filtration is P -complete, i.e. if $P[A] = 0$ for some $A \in \mathcal{F}$, then $A \in \mathcal{F}_0$.

Solution 10.1

- (a) Fix an $a > 0$ and a $t > 0$. We have to show that $\{\tau_a \leq t\} \in \mathcal{F}_t$. Observe that

$$\{\tau_a < t\} = \{W_s > a \text{ for some } s \in [0, t)\} = \{W_s > a \text{ for some } s \in [0, t]\}$$

because W has P -a.s. continuous trajectories. Note that we cannot have that

$$\{\tau_a \leq t\} = \{W_s > a \text{ for some } s \in [0, t]\}$$

because for a fixed $\omega \in \Omega$ it can happen that $\tau_a = t$ even if $W_s(\omega) \not> a$ for any $s \in [0, t]$. This is the case when $W_t(\omega) = a$ and $W_{t+\epsilon}(\omega) > a$ for all $\epsilon \in (0, \delta)$ for some $\delta > 0$. This also suggests why the right continuity of the filtration is needed.

Now, it is clear that we can write

$$\{\tau_a < t\} = \{W_s > a \text{ for some } s \in [0, t]\} = \bigcup_{s \in [0, t]} \{W_s > a\}.$$

However, since the set $[0, t]$ is an uncountable one, the union is uncountable as well, and since a σ -algebra is by definition closed only under at most *countable* unions and not necessarily

under *uncountable* unions, we cannot conclude. But for a fixed $\omega \in \Omega$ such that $t \mapsto W_t(\omega)$ is continuous, if $W_s(\omega) > a$, there exists some $\tilde{s} \in \mathbb{Q} \cap [0, t]$ such that $W_{\tilde{s}}(\omega) > a$. Hence

$$\{W_s > a \text{ for some } s \in [0, t]\} = \{W_s > a \text{ for some } s \in \mathbb{Q} \cap [0, t]\} = \bigcup_{s \in \mathbb{Q} \cap [0, t]} \{W_s > a\}.$$

Note that $\{W_s > a\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \in \mathbb{Q} \cap [0, t]$. Hence, since $\mathbb{Q} \cap [0, t]$ is now a countable set, we can conclude that $\bigcup_{s \in \mathbb{Q} \cap [0, t]} \{W_s > a\} \in \mathcal{F}_t$ and thus

$$\{\tau_a < t\} \in \mathcal{F}_t.$$

We can also express

$$\{\tau_a \leq t\} = \bigcap_{n=1}^{\infty} \{\tau_a < t + 1/n\} = \bigcap_{n=N}^{\infty} \{\tau_a < t + 1/n\} \quad \forall N \in \mathbb{N},$$

since $\{\tau_a < t + 1/(n+1)\} \subseteq \{\tau_a < t + 1/n\}$ for all $n \in \mathbb{N}$. Fix now an $N \in \mathbb{N}$ and observe that what we have shown before gives us that

$$\{\tau_a < t + 1/n\} \in \mathcal{F}_{t+1/n} \subseteq \mathcal{F}_{t+1/N}$$

for all $n \geq N$. Since σ -algebras are closed under countable intersections, we obtain that $\bigcap_{n=N}^{\infty} \{\tau_a < t + 1/n\} \in \mathcal{F}_{t+1/N}$ and thus $\{\tau_a \leq t\} \in \mathcal{F}_{t+1/N}$ for all $N \in \mathbb{N}$. Since the filtration is right-continuous, we can conclude that

$$\{\tau_a \leq t\} \in \bigcap_{N \in \mathbb{N}} \mathcal{F}_{t+1/N} = \mathcal{F}_t.$$

For the second part, fix $a_1, a_2 \in \mathbb{R}$ such that $a_1 < a_2$. Note that for all $\omega \in \Omega$

$$\{t \geq 0 \mid W_t(\omega) > a_2\} \subseteq \{t \geq 0 \mid W_t(\omega) > a_1\}$$

and thus

$$\tau_{a_2} = \inf\{t \geq 0 \mid W_t > a_2\} \geq \inf\{t \geq 0 \mid W_t > a_1\} = \tau_{a_1} \quad P\text{-a.s.}$$

- (b) By the global law of the iterated logarithm, we know that there exists an $\Omega_0 \subseteq \Omega$ such that $P[\Omega_0] = 1$ and

$$\limsup_{t \rightarrow \infty} \frac{W_t(\omega)}{\sqrt{2t \log(\log t)}} = 1 \quad \forall \omega \in \Omega_0. \tag{1}$$

Fix an $\omega \in \Omega_0$. Since $\lim_{t \rightarrow \infty} \sqrt{2t \log(\log t)} = \infty$, (1) implies that

$$\limsup_{t \rightarrow \infty} W_t(\omega) = \infty \quad \forall \omega \in \Omega_0.$$

Hence for all $x \in \mathbb{R}$ and $T > 0$, there exists a $t_x = t_x(\omega) > T$ such that $W_{t_x}(\omega) > x$. Choosing $x = a$, we see that $\tau_a(\omega) < t_x < \infty$. As a result $\tau_a(\omega) < \infty$ for all $\omega \in \Omega_0$ and thus

$$P[\tau_a < \infty] \geq P[\Omega_0] = 1.$$

- (c) Fix an $a > 0$ and recall that $W_0 = 0 < a$ P -a.s. and thus, by continuity of Brownian motion, $\tau_a > 0$ P -a.s. By definition of τ_a , we have for all $\omega \in \Omega$ that

$$W_t(\omega) \leq a \quad \forall t < \tau_a(\omega).$$

Since $\tau_a > 0$ P -a.s., the set $[0, \tau_a)$ is nonempty P -a.s. and hence, again by continuity of Brownian motion,

$$W_{\tau_a(\omega)}(\omega) = \lim_{t \uparrow \tau_a(\omega)} W_t(\omega) \leq a,$$

for P -almost all $\omega \in \Omega$, i.e. $W_{\tau_a} \leq a$ P -a.s.

Fix now an $\omega \in \Omega_0$. By the definition of an infimum, there is a sequence

$$(t_n)_{n \in \mathbb{N}} \subseteq \{t \geq 0 \mid W_t(\omega) > a\}$$

such that $\lim_{n \rightarrow \infty} t_n = \tau_a(\omega)$. Observe that $W_{t_n}(\omega) > a$ for all $n \in \mathbb{N}$ and thus, by continuity,

$$W_{\tau_a(\omega)}(\omega) = \lim_{n \rightarrow \infty} W_{t_n}(\omega) \geq a.$$

Since the Brownian motion is continuous P -a.s., we obtain that $W_{\tau_a} \geq a$ P -a.s., and we can conclude that $W_{\tau_a} = a$ P -a.s.

We can now conclude that

$$E[W_{\tau_{a_2}} \mid \mathcal{F}_{\tau_{a_1}}] = a_2 \neq a_1 = W_{\tau_{a_1}} \quad P\text{-a.s.}$$

We have thus proved that even for Brownian motion, a continuous (P, \mathbb{F}) -martingale, there exist \mathbb{F} -stopping times σ, τ with $\sigma \leq \tau$ such that

$$E[W_\tau \mid \mathcal{F}_\sigma] \neq W_\sigma \quad P\text{-a.s.}$$

- (d) As explained in (b), by the global law of the iterated logarithm there exists an $\Omega_0 \subseteq \Omega$ such that $P[\Omega_0] = 1$ and

$$\limsup_{t \rightarrow \infty} W_t(\omega) = \infty \quad \forall \omega \in \Omega_0.$$

Hence, for all $x \in \mathbb{R}$ and $T > 0$, there exists a $t_x = t_x(\omega) > T$ such that $W_{t_x}(\omega) > x$. Choosing $x = a$, we see that $\rho_a(\omega) \geq t_x > T$. As a result $\rho_a(\omega) = \infty$ for all $\omega \in \Omega_0$ and thus

$$P[\rho_a = \infty] \geq P[\Omega_0] = 1.$$

This directly implies that $P[\rho_a \leq t] = 0$ and thus, since \mathbb{F} is P -complete, $\{\rho_a \leq t\} \in \mathcal{F}_0 \subseteq \mathcal{F}_t$ for all $t \geq 0$, proving that ρ_a is a stopping time.

Exercise 10.2 Let M be an RCLL local martingale null at 0 which satisfies $\sup_{0 \leq s \leq T} |M_t| \in L^2$ for some $T \in \mathbb{R}$.

- (a) Show that M is a square-integrable martingale on $[0, T]$.
Hint: Dominated convergence theorem.

- (b) Let $[M]$ be the square bracket process of M . Show that

$$E[[M]_t - [M]_s \mid \mathcal{F}_s] = \text{Var}[M_t - M_s \mid \mathcal{F}_s]$$

for all $0 \leq s \leq t \leq T$.

Solution 10.2

- (a) Let us denote $M_T^* = \sup_{0 \leq u \leq T} |M_u|$. The process M is adapted by definition since it is a local martingale. We also have that $M_s \in L^2$ for all $s \in [0, T]$ since we clearly have that $|M_s|^2 \leq |M_T^*|^2$ for all $s \in [0, T]$ and $M_T^* \in L^2$ by assumption, so M is square-integrable on $[0, T]$.

Now let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence for M . We have for all $0 \leq s \leq t \leq T$ and all $n \in \mathbb{N}$ that

$$E[M_{\tau_n \wedge t} \mid \mathcal{F}_s] = M_{\tau_n \wedge s} \tag{2}$$

because M is a local martingale. Since $|M_{\tau_n \wedge t}|$ is bounded above for all $0 \leq t \leq T$ by an integrable random variable M_T^* , dominated convergence for conditional expectations gives us that

$$\lim_{n \rightarrow \infty} E[M_{\tau_n \wedge t} | \mathcal{F}_s] = E \left[\lim_{n \rightarrow \infty} M_{\tau_n \wedge t} \mid \mathcal{F}_s \right] = E[M_t | \mathcal{F}_s]. \quad (3)$$

On the other hand, we have for the right-hand side of (2) that

$$\lim_{n \rightarrow \infty} M_{\tau_n \wedge s} = M_s,$$

which together with (3) gives us the martingale property for M on $[0, T]$ and concludes the proof.

- (b) Since M is a square integrable martingale on $[0, T]$, the square bracket process $[M]$ is integrable and $M^2 - [M]$ is a martingale according to Theorem V.1.1 in the lecture notes. Therefore

$$\begin{aligned} E[[M]_t - [M]_s | \mathcal{F}_s] &= E[M_t^2 - M_s^2 | \mathcal{F}_s] \\ &= E[M_t^2 - 2M_tM_s + M_s^2 + 2M_tM_s - 2M_s^2 | \mathcal{F}_s] \\ &= E[(M_t - M_s)^2 | \mathcal{F}_s] + 2M_sE[M_t - M_s | \mathcal{F}_s] \\ &= E[(M_t - M_s)^2 | \mathcal{F}_s] = E[(M_t - E[M_t | \mathcal{F}_s])^2 | \mathcal{F}_s] \\ &= E[M_t - M_s + M_s - E[M_t | \mathcal{F}_s])^2 | \mathcal{F}_s] \\ &= E\left[\left((M_t - M_s) - E[M_t - M_s | \mathcal{F}_s]\right)^2 \mid \mathcal{F}_s\right] \\ &= \text{Var}[M_t - M_s | \mathcal{F}_s] \end{aligned}$$

for all $0 \leq s \leq t \leq T$.

Exercise 10.3 On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, consider an adapted stochastic process $X = (X_t)_{t \geq 0}$ null at 0. Assume that it is integrable and has independent stationary increments, i.e. $X_t - X_s$ is independent of \mathcal{F}_s and has the same distribution as X_{t-s} for all $t > s$. (In particular, this is satisfied for any Lévy process $L = (L_t)_{t \geq 0}$ with $E[|L_1|] < \infty$).

- (a) What conditions must $(E[X_t])_{t \geq 0}$ satisfy in order to make X a (P, \mathbb{F}) -supermartingale, a (P, \mathbb{F}) -submartingale, or a (P, \mathbb{F}) -martingale?
- (b) Assume from now on that X is a square-integrable (P, \mathbb{F}) -martingale. Prove that we have for all $t, s > 0$ that

$$E[X_t^2] + E[X_s^2] = E[X_{t+s}^2]$$

and deduce that $(E[X_t^2])_{t \geq 0}$ is an increasing process.

- (c) Use (b) to prove that $E[X_t^2] = tE[X_1^2]$ for all $t \geq 0$.
Hint: Prove the result first for $t = 1/n$ for all $n \in \mathbb{N}$. Deduce that it holds true for all $t \in \mathbb{Q}_+$ and use monotonicity to conclude.
- (d) Prove that $\langle X \rangle_t = tE[X_1^2]$, for all $t \geq 0$.
Hint: Use your result from (c).

Solution 10.3

- (a) Adaptedness and integrability are already given by assumption. We can then use that X has independent stationary increments to compute

$$E[X_t | \mathcal{F}_s] = E[X_t - X_s | \mathcal{F}_s] + X_s = E[X_t - X_s] + X_s = E[X_{t-s}] + X_s,$$

for all $t > s$. As a result X is a (P, \mathbb{F}) -supermartingale if and only if $E[X_t] \leq 0$ for all $t \geq 0$, a (P, \mathbb{F}) -submartingale if and only if $E[X_t] \geq 0$ for all $t \geq 0$, and a (P, \mathbb{F}) -martingale if and only if $E[X_t] = 0$ for all $t \geq 0$.

- (b) By the martingale property of X and the stationarity of the increments, we can directly compute

$$\begin{aligned} E[X_{t+s}^2] - E[X_t^2] &= E[X_{t+s}^2 - X_t^2] = E[(X_{t+s} - X_t)^2] + 2E[X_t(X_{t+s} - X_t)] \\ &= E[(X_{t+s} - X_t)^2] + 2E[X_t E[X_{t+s} - X_t | \mathcal{F}_t]] = E[X_s^2], \end{aligned}$$

for all $t, s > 0$. As a result, for $t > s$ we have that

$$E[X_t^2] - E[X_s^2] = E[X_{t-s}^2] \geq 0,$$

proving that the process $(E[X_t^2])_{t \geq 0}$ is increasing.

- (c) Let first $t = 1/n$ for some $n \in \mathbb{N}$. We want to show that $nE[X_{1/n}^2] = E[X_1^2]$. We compute

$$nE[X_{1/n}^2] = \sum_{k=1}^n E[X_{1/n}^2] = \sum_{k=1}^n (E[X_{k/n}^2] - E[X_{(k-1)/n}^2]) = E[X_1^2] - E[X_0^2] = E[X_1^2],$$

where in the second equality we have used our result from (b). If we now consider an arbitrary number $\ell/n \in \mathbb{Q}_+$, we can use the same technique as in the above to compute

$$\ell E[X_{1/n}^2] = \sum_{k=1}^{\ell} E[X_{1/n}^2] = E[X_{\ell/n}^2].$$

Since we have that $E[X_{1/n}^2] = \frac{1}{n}E[X_1^2]$, we can conclude that $E[X_{\ell/n}^2] = \frac{\ell}{n}E[X_1^2]$. We have thus proved that $E[X_t^2] = tE[X_1^2]$ for all $t \in \mathbb{Q}_+$, but since $E[X_t^2]$ is increasing, we can compute

$$tE[X_1^2] = \sup_{s \in \mathbb{Q}_+, s < t} sE[X_1^2] = \sup_{s \in \mathbb{Q}_+, s < t} E[X_s^2] \leq E[X_t^2] \leq \inf_{s \in \mathbb{Q}_+, s > t} E[X_s^2] = tE[X_1^2],$$

and thus conclude that $E[X_t^2] = tE[X_1^2]$.

- (d) We first have to show that $(tE[X_1^2])_{t \geq 0}$ is an increasing, predictable process null at 0. Since $E[X_1^2] \geq 0$, the process is clearly increasing. Moreover, since it is deterministic, it is also predictable and it is clearly null at 0. It only remains to show that the process $(X_t^2 - tE[X_1^2])_{t \geq 0}$ is a (P, \mathbb{F}) -martingale. Since the increments are independent and stationary, for all $t > s$ we can compute

$$\begin{aligned} E[X_t^2 - X_s^2 | \mathcal{F}_s] &= E[(X_t - X_s)^2 | \mathcal{F}_s] + 2X_s E[X_t - X_s | \mathcal{F}_s] \\ &= E[(X_t - X_s)^2] = E[X_{t-s}^2] = (t-s)E[X_1^2] \\ &= tE[X_1^2] - sE[X_1^2], \end{aligned}$$

where the fourth equality uses our result from (c). Rearranging the above, we obtain that

$$E[X_t^2 - tE[X_1^2] - (X_s^2 - sE[X_1^2]) | \mathcal{F}_s] = 0,$$

which is the martingale property for $(X_t^2 - tE[X_1^2])_{t \geq 0}$.