Coordinator
Bálint Gersey

Mathematical Foundations for Finance

Exercise sheet 11

Please upload your solutions until Wednesday, 08/12/2021, 12:00 using the link on the course website.

Exercise 11.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Assume that $\mathcal{F}_0$ is P-trivial and consider a Brownian motion W on this space.

(a) Prove that any continuous, adapted process H is predictable and locally bounded.
Hint 1: Recall that a process X is locally bounded if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to infinity such that each X^{τ_n} is uniformly bounded P-a.s.

(b) Prove that any predictable, locally bounded process H is an element of $L^2_{\text{loc}}(W)$.
Hint: We saw that $L^2_{loc}(M)$ can be characterized in a nice way when M is a continuous local martingale null at 0.

(c) Deduce that for any function $f : \mathbb{R} \to \mathbb{R}$ in C^1, the stochastic integral $\int_0^\cdot f'(W_s)dW_s$ is a continuous local martingale.

(d) Conclude using Itô's formula that $f(W)$ for a given $f \in C^2$ is a continuous local martingale if and only if $\int_0^\cdot f''(W_s)ds = 0$.
Hint 1: If M and N are local $(P, \mathbb{F})$-martingales, then $M + N$ is a local $(P, \mathbb{F})$-martingale.
Hint 2: For every continuous local martingale M null at 0 and with finite variation, we have that $M = 0$ P-a.s.

Exercise 11.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion with respect to a probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Using Itô's formula, say and justify for each of the following processes whether they are local $(P, \mathbb{F})$-martingales or not. Which of them are even $(P, \mathbb{F})$-martingales?

(a) $X^{(1)}_t := \exp\left(\frac{1}{2}\alpha^2 t\right)\cos\left(\alpha(W_t - \beta)\right)$, $t \geq 0$, where $\alpha, \beta \in \mathbb{R}$.
Hint: For the martingale property of $X^{(1)}$, look first at $[0, T]$ for some $T > 0$.

(b) $X^{(2)}_t := \sin W_t - \cos W_t$, $t \geq 0$.

(c) $X^{(3)}_t := W^p_t - ptW_t$, $t \geq 0$, for $p \in \mathbb{N}$ with $p \geq 2$.
Hint: For any $T > 0$, $\sup_{0\leq t\leq T} W_t$ has the same distribution as $|W_T|$ and so has $-\inf_{0\leq s\leq T} W_s$.

Exercise 11.3 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion with respect to some probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Use Itô's formula to write the following processes as stochastic integrals.

(a) $X^{(1)}_t = W^2_t$.

(b) $X^{(2)}_t = t^2 W^3_t$.

(c) $X^{(3)}_t = \exp(mt + \sigma W_t)$.

(d) $X^{(4)}_t = \cos(t + W_t)$.

(e) $X^{(5)}_t = \log(2 + \cos(W_t - t))$.

(f) Let X and Y be two continuous real-valued $(P, \mathbb{F})$-semimartingales. Define the process $Z = XY$. Apply Itô's formula to Z and write it as a sum of stochastic integrals.