Mathematical Foundations for Finance

Exercise sheet 12

Please upload your solutions until Wednesday, 08/12/2021, 12:00 using the link on the course website.

Exercise 12.1 In this exercise, we show various results that are frequently used in stochastic analysis. Some of them were given as hints in the previous exercises.

(a) Let X be an RCLL $\mathbb{F}$-adapted stochastic process and τ an $\mathbb{F}$-stopping time. Show that if X^τ is an $\mathbb{F}$-martingale, then so is X^σ for any $\mathbb{F}$-stopping time σ with $\sigma \leq \tau$ P-a.s.
Hint: You can use the result that a stopped RCLL martingale is again an RCLL martingale. This is similar to the result you have proved in Exercise 3.1 (c).

(b) Let M and N be two RCLL local $\mathbb{F}$-martingales. Show that the linear combination $\alpha M + \beta N$ for any $\alpha, \beta \in \mathbb{R}$ is an RCLL local $\mathbb{F}$-martingale as well.
Hint: Make use of the result in (a).

(c) We say that two Brownian motions W^1 and W^2 on the same probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ are *correlated with correlation* $\rho \in [-1, 1]$ if for $s \leq t$, the increments $W^1_t - W^1_s$ and $W^2_t - W^2_s$ are independent of $\mathcal{F}_s$ and jointly normally distributed with $\mathcal{N}(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} t-s & \rho(t-s) \\ \rho(t-s) & t-s \end{pmatrix}.$$

Show that $[W^1, W^2]_t = \rho t$ P-a.s.
Hint: Define $B^\lambda = \lambda(W^1 + W^2)$ with $\lambda \in \mathbb{R}$. Find λ such that B^λ becomes a $(P, \mathbb{F})$-Brownian motion. Then compute $[B^\lambda]$ in terms of W^1 and W^2, using the properties of $[\,\cdot\,,\cdot\,]$.

Exercise 12.2 Let $X = (X_t)_{t\geq 0}$ be a continuous semimartingale null at 0. We define the process

$$L := \mathcal{E}(X) := e^{X - \frac{1}{2}[X]}.$$

(a) Show via Itô's formula that

$$L_t = 1 + \int_0^t L_s dX_s, \quad \forall t \geq 0. \tag{1}$$

Conclude that L is a continuous local martingale if and only if X is a continuous local martingale.

(b) Show that $L = \mathcal{E}(X)$ is the only solution to (1) for a given X.
Hint: Let L' be another solution of (1). Compute $\frac{L'}{L}$ using Itô's formula.

(c) Let $Y = (Y_t)_{t\geq 0}$ be another continuous semimartingale null at 0. Show *Yor's formula*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}\left(X + Y + [X, Y]\right).$$

Exercise 12.3 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a filtered probability space and consider two *independent* Brownian motions $W^1 = (W^1_t)_{t\in[0,T]}$ and $W^2 = (W^2_t)_{t\in[0,T]}$. Let $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ and $\widetilde{S}^2 = (\widetilde{S}^2_t)_{t\in[0,T]}$ be two *undiscounted* stock price processes with the dynamics

$$d\widetilde{S}^1_t = \widetilde{S}^1_t \left(\mu_1 dt + \sigma_1 dB^1_t\right), \quad \widetilde{S}^1_0 > 0,$$
$$d\widetilde{S}^2_t = \widetilde{S}^2_t \left(\mu_2 dt + \sigma_2 dB^2_t\right), \quad \widetilde{S}^2_0 > 0,$$

where $B^1 = W^1$, $B^2 = \alpha W^1 + \sqrt{1-\alpha^2}W^2$, for some $\alpha \in [0,1)$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$.

(a) Find the SDEs satisfied by $X^1 := \frac{\widetilde{S}^2}{\widetilde{S}^1}$ and $X^2 := \frac{\widetilde{S}^1}{\widetilde{S}^2}$.
Remark: Since $\widetilde{S}^1$ and $\widetilde{S}^2$ have continuous trajectories and satisfy $\widetilde{S}^1_t, \widetilde{S}^2_t > 0$ for all $t \in [0,T]$ P-a.s., we can choose each of them as *numéraire*.

(b) For $\beta_1, \beta_2 \in \mathbb{R}$, define the continuous local $(P,\mathbb{F})$-martingale $L^{(\beta_1,\beta_2)} := \beta_1 W^1 + \beta_2 W^2$. Show that for all $\beta_1, \beta_2 \in \mathbb{R}$, the stochastic exponential $Z^{(\beta_1,\beta_2)} := \mathcal{E}(L^{(\beta_1,\beta_2)})$ is a true $(P,\mathbb{F})$-martingale on $[0,T]$.

(c) For $\beta_1, \beta_2 \in \mathbb{R}$, define by $dQ^{(\beta_1,\beta_2)} = Z_T^{(\beta_1,\beta_2)} dP$ a probability measure $Q^{(\beta_1,\beta_2)}$ which is equivalent to P on $\mathcal{F}_T$. Fix $\beta_1, \beta_2 \in \mathbb{R}$. Using Girsanov's theorem, show that the two processes $\widetilde{W}^1_t := W^1_t - \beta_1 t$ and $\widetilde{W}^2_t := W^2_t - \beta_2 t$, $t \in [0,T]$, are local $(Q^{(\beta_1,\beta_2)},\mathbb{F})$-martingales. Conclude that
$$\widetilde{B}^1 := \widetilde{W}^1 \quad \text{and} \quad \widetilde{B}^2_t := B^2_t - (\alpha\beta_1 + \sqrt{1-\alpha^2}\beta_2)t, \quad t \in [0,T],$$
are local $(Q^{(\beta_1,\beta_2)},\mathbb{F})$-martingales as well.
Remark: One can show that $\widetilde{W}^1$ and $\widetilde{W}^2$ are *independent* Brownian motions under $Q^{(\beta_1,\beta_2)}$ and correspondingly that $\widetilde{B}^1$ and $\widetilde{B}^2$ are *correlated* Brownian motions under $Q^{(\beta_1,\beta_2)}$.

(d) What conditions on $\beta_1, \beta_2 \in \mathbb{R}$ make the processes X^1 and X^2 $(Q^{(\beta_1,\beta_2)},\mathbb{F})$-martingales?