Mathematical Foundations for Finance

Exercise sheet 12

Please upload your solutions until Wednesday, 08/12/2021, 12:00 using the link on the course website.

Exercise 12.1 In this exercise, we show various results that are frequently used in stochastic analysis. Some of them were given as hints in the previous exercises.

(a) Let X be an RCLL $\mathbb{F}$-adapted stochastic process and τ an $\mathbb{F}$-stopping time. Show that if X^τ is an $\mathbb{F}$-martingale, then so is X^σ for any $\mathbb{F}$-stopping time σ with $\sigma \leq \tau$ P-a.s.
Hint: You can use the result that a stopped RCLL martingale is again an RCLL martingale. This is similar to the result you have proved in Exercise 3.1 (c).

(b) Let M and N be two RCLL local $\mathbb{F}$-martingales. Show that the linear combination $\alpha M + \beta N$ for any $\alpha, \beta \in \mathbb{R}$ is an RCLL local $\mathbb{F}$-martingale as well.
Hint: Make use of the result in (a).

(c) We say that two Brownian motions W^1 and W^2 on the same probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ are *correlated with correlation* $\rho \in [-1, 1]$ if for $s \leq t$, the increments $W^1_t - W^1_s$ and $W^2_t - W^2_s$ are independent of $\mathcal{F}_s$ and jointly normally distributed with $\mathcal{N}(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} t-s & \rho(t-s) \\ \rho(t-s) & t-s \end{pmatrix}.$$

Show that $[W^1, W^2]_t = \rho t$ P-a.s.
Hint: Define $B^\lambda = \lambda(W^1 + W^2)$ with $\lambda \in \mathbb{R}$. Find λ such that B^λ becomes a $(P, \mathbb{F})$-Brownian motion. Then compute $[B^\lambda]$ in terms of W^1 and W^2, using the properties of $[\,\cdot\,,\,\cdot\,]$.

Solution 12.1

(a) For notational clarity, let us denote $Y = X^\tau$. Note that since $\sigma \leq \tau$ P-a.s. by assumption, we can write for all $t \geq 0$ that

$$X^\sigma_t = X_{t\wedge\sigma} = X_{t\wedge\tau\wedge\sigma} = X^\tau_{t\wedge\sigma} = Y^\sigma_t \quad P\text{-a.s.}$$

But Y is an RCLL $\mathbb{F}$-martingale by assumption, so Y^σ, the stopped martingale, is an RCLL $\mathbb{F}$-martingale as well. The above equation then directly implies the same for X^σ.

(b) Let $(\tau_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ be two localizing sequences for M and N, respectively, and let $\theta_n := \min(\tau_n, \sigma_n)$. Since we have by the definition of θ_n that $\theta_n \leq \tau_n$ P-a.s. as well as $\theta_n \leq \sigma_n$ P-a.s., our result from (a) implies that if θ_n is a stopping time, then both M^{θ_n} and N^{θ_n} are RCLL $\mathbb{F}$-martingales for all $n \in \mathbb{N}$ because M^{τ_n} and N^{σ_n} are RCLL local $\mathbb{F}$-martingales for all $n \in \mathbb{N}$. However, since linear combinations of RCLL $\mathbb{F}$-martingales are easily seen to be RCLL $\mathbb{F}$-martingales, we also have that

$$\alpha M^{\theta_n} + \beta N^{\theta_n} = (\alpha M + \beta N)^{\theta_n}$$

is an RCLL $\mathbb{F}$-martingale.

What thus remains to be shown is that $(\theta_n)_{n\in\mathbb{N}}$ is indeed a sequence of stopping times with $\theta_n \nearrow \infty$ P-a.s. The part that $\theta_n \nearrow \infty$ P-a.s. is obvious, since we have that both $\tau_n \nearrow \infty$ and $\sigma \nearrow \infty$ P-a.s. In order to show that θ_n is a stopping time for each $n \in \mathbb{N}$, note that

$$\begin{aligned}\{\theta_n \le t\} &= \{\min(\tau_n, \sigma_n) \le t\} = \{\omega \in \Omega \text{ such that } \tau_n(\omega) \le t \text{ or } \sigma_n(\omega) \le t\}\\ &= \{\tau_n \le t\} \cup \{\sigma_n \le t\} \in \mathcal{F}_t,\end{aligned}$$

since $\{\tau_n \le t\} \in \mathcal{F}_t$ and $\{\sigma_n \le t\} \in \mathcal{F}_t$ because τ_n and σ_n are $\mathbb{F}$-stopping times and $\mathcal{F}_t$ is as a σ-algebra closed under finite unions. This shows that $(\theta_n)_{n\in\mathbb{N}}$ is indeed a localizing sequence for $\alpha M + \beta N$ and concludes the proof.

(c) Since W^1 and W^2 are $(P,\mathbb{F})$-Brownian motions, $(W^1_t - W^1_s, W^2_t - W^2_s) \sim \mathcal{N}_s(\mu, \Sigma)$ with

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} t-s & \rho(t-s) \\ \rho(t-s) & t-s \end{pmatrix}$$

and we know that linear transformations of normal random vectors are normally distributed, it should be possible to select λ such that B^λ is a $(P,\mathbb{F})$-Brownian motion, in which case we know that $[B^\lambda]_t = t$.

Indeed, we clearly have $B^\lambda_0 = 0$ P-a.s., continuous trajectories for P-a.a. $\omega \in \Omega$ as well as $\mathbb{F}$-adaptedness. We therefore only need to choose $\lambda \in \mathbb{R}$ such that for all $s \le t$ we have that $B^\lambda_t - B^\lambda_s$ is independent of $\mathcal{F}_s$ and has a normal distribution $\mathcal{N}(0, t-s)$. We have that

$$\begin{aligned}B^\lambda_t - B^\lambda_s = \lambda\left(W^1_t - W^1_s\right) + \lambda\left(W^2_t - W^2_s\right) &\sim \mathcal{N}\big(0, \lambda^2(2(t-s) + 2\rho(t-s)\big)\\ &\sim \mathcal{N}\big(0, \lambda^2(t-s)(2+2\rho)\big).\end{aligned}$$

Setting $\lambda^2 = 1/(2+2\rho)$ thus leads to B^λ being a $(P,\mathbb{F})$-Brownian motion.

Now, since we have defined the covariation $[M,N]$ of two local martingales M and N null at 0 using polarization, i.e.

$$[M,N] := \frac{1}{4}\left([M+N] - [M-N]\right),$$

we obtain that $[M,M] = [M]$ P-a.s. As suggested in the hint, let us now compute the quadratic variation of $B^\lambda = \lambda\left(W^1 + W^2\right)$ using that $[B^\lambda] = [B^\lambda, B^\lambda]$ and the bilinearity and symmetry of $[\,\cdot\,,\,\cdot\,]$. We compute

$$\begin{aligned}[B^\lambda]_t &= [\lambda\left(W^1 + W^2\right), \lambda\left(W^1 + W^2\right)]_t = \lambda^2[W^1 + W^2, W^1 + W^2]_t\\ &= \lambda^2\left([W^1, W^1 + W^2]_t + [W^2, W^1 + W^2]_t\right)\\ &= \lambda^2\left([W^1, W^1]_t + [W^1, W^2]_t + [W^2, W^1]_t + [W^2, W^2]_t\right)\\ &= \lambda^2\left([W^1]_t + 2[W^1, W^2] + [W^2]_t\right)\\ &= 2\lambda^2\left([W^1, W^2]_t + t\right),\end{aligned}$$

where the last equality follows from the fact that W^1 and W^2 are $(P,\mathbb{F})$-Brownian motions, and we thus have that $[W^1]_t = t$ and $[W^2]_t = t$ P-a.s. We can thus rearrange the above to obtain that

$$[W^1, W^2]_t = \frac{1}{2\lambda^2}[B^\lambda]_t - t. \tag{1}$$

But we already know that setting $\lambda^2 = 1/(2+2\rho)$ leads to B^λ being a $(P,\mathbb{F})$-Brownian motion, in which case $[B^\lambda]_t = t$ P-a.s. Plugging these into (1) yields

$$[W^1, W^2]_t = t\left(\frac{1}{2\lambda^2} - 1\right) = t\left(\frac{2+2\rho}{2} - 1\right) = \rho t \quad P\text{-a.s. for all } t \ge 0.$$

Alternatively, similarly to $[W^1]_t$ and $[W^2]_t$, it is also possible to obtain $[W^1, W^2]_t$ as a limit over some fixed sequence of refining partitions, more specifically

$$P\left[\lim_{n\to\infty}\sum_{t_i\in\Pi_n}\left(W^1_{t_{i+1}\wedge t}-W^1_{t_i\wedge t}\right)\left(W^2_{t_{i+1}\wedge t}-W^2_{t_i\wedge t}\right)=[W^1,W^2]_t \text{ for all } t\geq 0\right]=1.$$

Using this expression as a limit of some sequence of refining partitions, one can thus show that

$$E\left[\left([W^1,W^2]_t-\rho t\right)^2\right]=0,$$

which then also implies the result (as seen for instance in Exercise 1.2).

Exercise 12.2 Let $X=(X_t)_{t\geq 0}$ be a continuous semimartingale null at 0. We define the process

$$L:=\mathcal{E}(X):=e^{X-\frac{1}{2}[X]}.$$

(a) Show via Itô's formula that

$$L_t=1+\int_0^t L_s dX_s,\quad \forall t\geq 0. \tag{2}$$

Conclude that L is a continuous local martingale if and only if X is a continuous local martingale.

(b) Show that $L=\mathcal{E}(X)$ is the only solution to (2) for a given X.
Hint: Let L' be another solution of (2). Compute $\frac{L'}{L}$ using Itô's formula.

(c) Let $Y=(Y_t)_{t\geq 0}$ be another continuous semimartingale null at 0. Show *Yor's formula*

$$\mathcal{E}(X)\mathcal{E}(Y)=\mathcal{E}\left(X+Y+[X,Y]\right).$$

Solution 12.2

(a) We apply Itô's formula to the C^2 function $f(x,y):=e^{x-\frac{1}{2}y}$ and the semimartingale $(X_t,[X]_t)_{t\geq 0}$. We obtain (omitting the subscripts for the time parameter t) that

$$\begin{aligned}dL&=df(X,[X])\\&=\frac{\partial}{\partial x}f(X,[X])dX+\frac{\partial}{\partial y}f(X,[X])d[X]+\frac{1}{2}\frac{\partial^2}{\partial x^2}f(X,[X])d[X]\\&\quad+\frac{1}{2}\frac{\partial^2}{\partial y^2}f(X,[X])d[[X]]+\frac{\partial^2}{\partial x\partial y}f(X,[X])d[X,[X]].\end{aligned}$$

However, since X is continuous and $[X]$ is continuous and of finite variation, we have that $[[X]]\equiv 0$ and $[X,[X]]\equiv 0$, so the last two terms disappear. A direct computation shows that $\frac{\partial}{\partial y}f+\frac{1}{2}\frac{\partial^2}{\partial x^2}f=0$ and $\frac{\partial}{\partial x}f=f$. Inserting these relations into the equation above yields

$$dL=LdX,\quad \text{or}\quad L_t=1+\int_0^t L_s dX_s.$$

As the above C^2-transformation of the continuous semimartingale $(X_t,[X]_t)_{t\geq 0}$, L is always a continuous semimartingale (hence predictable and locally bounded). Therefore, $L\in L^2_{\mathrm{loc}}(M)$ for all continuous local martingales M. If X is a continuous local martingale, then we may choose $M=X$ to conclude that L is a continuous local martingale.

Conversely, since L is strictly positive by definition, X is given by

$$dX = \frac{1}{L} dL \quad \text{or} \quad X_t = \int_0^t \frac{1}{L_s} dL_s.$$

Therefore, if L is a continuous local martingale, then X is a local martingale by the same reasoning as above.

(b) Let L' be another process such that

$$dL' = L' dX, \quad L'_0 = 1.$$

Since such an L' is necessarily a semimartingale, we can apply Itô's formula to the quotient $\frac{L'}{L} = f(L', L)$ with the function $f(x, y) = \frac{x}{y}$. A direct computation yields

$$\frac{\partial}{\partial x} f(x,y) = \frac{1}{y}, \quad \frac{\partial}{\partial y} f(x,y) = -\frac{x}{y^2}, \quad \frac{\partial^2}{\partial x^2} f(x,y) = 0,$$
$$\frac{\partial^2}{\partial x \partial y} f(x,y) = -\frac{1}{y^2}, \quad \frac{\partial^2}{\partial y^2} f(x,y) = 2\frac{x}{y^3}.$$

Plugging these into Itô's formula and using that $dL = LdX$ and $dL' = L'dX$ gives that $d[L] = L^2 d[X]$, $d[L', L] = L'Ld[X]$ which then yields (again omitting the time parameter)

$$\begin{aligned} d\left(\frac{L'}{L}\right) &= \frac{1}{L} dL' - \frac{L'}{L^2} dL - \frac{1}{L^2} d[L', L] + \frac{L'}{L^3} d[L] \\ &= \frac{L'}{L} dX - \frac{L'}{L} dX - \frac{L'}{L} d[X] + \frac{L'}{L} d[X] \\ &= 0. \end{aligned}$$

Hence, we conclude that $\frac{L'_t}{L_t} = 1$ P-a.s. for all $t \geq 0$.

(c) One solution which does not use the explicit formula for $\mathcal{E}(X)$ uses the product rule and the fact that $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are the unique solutions of $dL = LdX$ and $dL = LdY$, respectively. We then find that

$$\begin{aligned} d\left(\mathcal{E}(X)\mathcal{E}(Y)\right) &= \mathcal{E}(X) d\mathcal{E}(Y) + \mathcal{E}(Y) d\mathcal{E}(X) + d[\mathcal{E}(X), \mathcal{E}(Y)] \\ &= \mathcal{E}(X)\mathcal{E}(Y) dY + \mathcal{E}(Y)\mathcal{E}(X) dX + \mathcal{E}(X)\mathcal{E}(Y) d[X, Y] \\ &= \mathcal{E}(X)\mathcal{E}(Y) d\left(X + Y + [X, Y]\right). \end{aligned}$$

By the uniqueness of the solution to $dL = LdX$ for X replaced by $X + Y + [X, Y]$, we conclude that

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]).$$

Alternatively, one can use the explicit formula for the stochastic exponential and compute.

Exercise 12.3 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be a filtered probability space and consider two *independent* Brownian motions $W^1 = (W_t^1)_{t \in [0,T]}$ and $W^2 = (W_t^2)_{t \in [0,T]}$. Let $\widetilde{S}^1 = (\widetilde{S}_t^1)_{t \in [0,T]}$ and $\widetilde{S}^2 = (\widetilde{S}_t^2)_{t \in [0,T]}$ be two *undiscounted* stock price processes with the dynamics

$$\begin{aligned} d\widetilde{S}_t^1 &= \widetilde{S}_t^1 \left(\mu_1 dt + \sigma_1 dB_t^1\right), \quad \widetilde{S}_0^1 > 0, \\ d\widetilde{S}_t^2 &= \widetilde{S}_t^2 \left(\mu_2 dt + \sigma_2 dB_t^2\right), \quad \widetilde{S}_0^2 > 0, \end{aligned}$$

where $B^1 = W^1$, $B^2 = \alpha W^1 + \sqrt{1 - \alpha^2} W^2$, for some $\alpha \in [0, 1)$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$.

(a) Find the SDEs satisfied by $X^1 := \frac{\widetilde{S}^2}{\widetilde{S}^1}$ and $X^2 := \frac{\widetilde{S}^1}{\widetilde{S}^2}$.
Remark: Since $\widetilde{S}^1$ and $\widetilde{S}^2$ have continuous trajectories and satisfy $\widetilde{S}^1_t, \widetilde{S}^2_t > 0$ for all $t \in [0,T]$ P-a.s., we can choose each of them as *numéraire*.

(b) For $\beta_1, \beta_2 \in \mathbb{R}$, define the continuous local $(P, \mathbb{F})$-martingale $L^{(\beta_1,\beta_2)} := \beta_1 W^1 + \beta_2 W^2$. Show that for all $\beta_1, \beta_2 \in \mathbb{R}$, the stochastic exponential $Z^{(\beta_1,\beta_2)} := \mathcal{E}(L^{(\beta_1,\beta_2)})$ is a true $(P, \mathbb{F})$-martingale on $[0,T]$.

(c) For $\beta_1, \beta_2 \in \mathbb{R}$, define by $dQ^{(\beta_1,\beta_2)} = Z_T^{(\beta_1,\beta_2)} dP$ a probability measure $Q^{(\beta_1,\beta_2)}$ which is equivalent to P on $\mathcal{F}_T$. Fix $\beta_1, \beta_2 \in \mathbb{R}$. Using Girsanov's theorem, show that the two processes $\widetilde{W}^1_t := W^1_t - \beta_1 t$ and $\widetilde{W}^2_t := W^2_t - \beta_2 t$, $t \in [0,T]$, are local $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingales. Conclude that

$$\widetilde{B}^1 := \widetilde{W}^1 \quad \text{and} \quad \widetilde{B}^2_t := B^2_t - (\alpha\beta_1 + \sqrt{1-\alpha^2}\beta_2)t, \quad t \in [0,T],$$

are local $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingales as well.
Remark: One can show that $\widetilde{W}^1$ and $\widetilde{W}^2$ are *independent* Brownian motions under $Q^{(\beta_1,\beta_2)}$ and correspondingly that $\widetilde{B}^1$ and $\widetilde{B}^2$ are *correlated* Brownian motions under $Q^{(\beta_1,\beta_2)}$.

(d) What conditions on $\beta_1, \beta_2 \in \mathbb{R}$ make the processes X^1 and X^2 $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingales?

Solution 12.3

(a) Take $i \neq j$. By Itô's formula, we get

$$\begin{aligned} dX^i = d\left(\frac{\widetilde{S}^j}{\widetilde{S}^i}\right) &= \frac{1}{\widetilde{S}^i} d\widetilde{S}^j - \frac{\widetilde{S}^j}{(\widetilde{S}^i)^2} d\widetilde{S}^i - \frac{1}{(\widetilde{S}^i)^2} d[\widetilde{S}^i, \widetilde{S}^j] + \frac{\widetilde{S}^j}{(\widetilde{S}^i)^3} d[\widetilde{S}^i] \\ &= X^i \left((\mu_j - \mu_i + \sigma_i^2 - \alpha\sigma_i\sigma_j)dt + \sigma_j dB^j - \sigma_i dB^i \right). \end{aligned}$$

(b) Fix $\beta_1, \beta_2 \in \mathbb{R}$. Then clearly $L^{(\beta_1,\beta_2)}$ is a $(P, \mathbb{F})$-martingale, whose quadratic variation satisfies P-a.s. for all $t \in [0,T]$

$$[L^{(\beta_1,\beta_2)}]_t = [\beta_1 W^1 + \beta_2 W^2]_t = \beta_1^2 t + \beta_2^2 t,$$

where we used that $[W^1, W^2] = 0$, as shown in Exercise 12.1 (c). Moreover, by independence of W^1 and W^2 and Proposition IV.2.2 in the lecture notes, we have

$$\begin{aligned} E\left[\frac{Z_t^{(\beta_1,\beta_2)}}{Z_s^{(\beta_1,\beta_2)}} \,\middle|\, \mathcal{F}_s \right] &= E\left[\frac{e^{\beta_1 W^1_t + \beta_2 W^2_t - \frac{1}{2}(\beta_1^2+\beta_2^2)t}}{e^{\beta_1 W^1_s + \beta_2 W^2_s - \frac{1}{2}(\beta_1^2+\beta_2^2)s}} \,\middle|\, \mathcal{F}_s \right] \\ &= E\left[e^{\beta_1(W^1_t - W^1_s) + \beta_2(W^2_t - W^2_s) - \frac{1}{2}(\beta_1^2+\beta_2^2)(t-s)} \,\middle|\, \mathcal{F}_s \right] \\ &= e^{-\frac{1}{2}(\beta_1^2+\beta_2^2)(t-s)} E\left[e^{\beta_1(W^1_t - W^1_s) + \beta_2(W^2_t - W^2_s)} \,\middle|\, \mathcal{F}_s \right] \\ &= e^{-\frac{1}{2}\beta_1^2(t-s)} E\left[e^{\beta_1(W^1_t - W^1_s)} \right] e^{-\frac{1}{2}\beta_2^2(t-s)} E\left[e^{\beta_2(W^2_t - W^2_s)} \right] \\ &= 1, \end{aligned}$$

so $Z^{(\beta_1,\beta_2)}$ has the martingale property. Adaptedness is clear and the integrability follows from the fact that $Z_t^{(\beta_1,\beta_2)}$ is as an exponential of a normally distributed random variable log-normally distributed for all $t \geq 0$, and we know that all moments of log-normal distribution are finite. Therefore, $Z_t^{(\beta_1,\beta_2)}$ is a $(P, \mathbb{F})$-martingale.

(c) By Girsanov's theorem in the form of Theorem VI.2.3 in the lecture notes, we know that

$$\widetilde{W}^1 = W^1 - [L^{(\beta_1,\beta_2)}, W^1] \quad \text{and} \quad \widetilde{W}^2 = W^2 - [L^{(\beta_1,\beta_2)}, W^2]$$

are local $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingales. Thus, it suffices to show that we have P-a.s. for all $t \in [0,T]$

$$[L^{(\beta_1,\beta_2)}, W^1]_t = \beta_1 t \quad \text{and} \quad [L^{(\beta_1,\beta_2)}, W^2]_t = \beta_2 t.$$

But this follows immediately from independence of W^1 and W^2 and the definition of $L^{(\beta_1,\beta_2)}$.

To conclude the final part, we simply write out the definition of the corresponding processes to get

$$\begin{aligned}\widetilde{B}^2_t &= B^2_t - (\alpha\beta_1 + \sqrt{1-\alpha^2}\beta_2)t = \alpha(W^1_t - \beta_1 t) + \sqrt{1-\alpha^2}(W^2_t - \beta_2 t) \\ &= \alpha\widetilde{W}^1_t + \sqrt{1-\alpha^2}\widetilde{W}^2_t,\end{aligned} \tag{3}$$

which is a linear combination of local $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingales.

(d) First, we note that X^1 and X^2 still satisfy the same SDEs under $Q^{(\beta_1,\beta_2)}$ with the only difference that B^1 and B^2 are in general no longer Brownian motions under $Q^{(\beta_1,\beta_2)}$. Using that $\widetilde{B}^1$ and $\widetilde{B}^2$ are local martingales under $Q^{(\beta_1,\beta_2)}$, we get by (a) that

$$\begin{aligned}dX^i &= X^i\left((\mu_j - \mu_i + \sigma_i^2 - \alpha\sigma_i\sigma_j)dt + \sigma_j d(\widetilde{B}^j + \gamma_j t) - \sigma_i d(\widetilde{B}^i + \gamma_i t)\right) \\ &= X^i\left((\mu_j - \mu_i + \sigma_i^2 - \alpha\sigma_i\sigma_j + \sigma_j\gamma_j - \sigma_i\gamma_i)dt + \sigma_j d\widetilde{B}^j - \sigma_i d\widetilde{B}^i\right),\end{aligned} \tag{4}$$

where $\gamma_1 := \beta_1$ and $\gamma_2 := \alpha\beta_1 + \sqrt{1-\alpha^2}\beta_2$. Next, X^i is a local $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingale if and only if the drift component in (4) vanishes, i.e.,

$$\mu_j - \mu_i + \sigma_i^2 - \alpha\sigma_i\sigma_j + \sigma_j\gamma_j - \sigma_i\gamma_i = 0 \tag{5}$$

and we express the local martingale components in terms of the independent Brownian motions $\widetilde{W}^1$ and $\widetilde{W}^2$ (c.f. (3)),

$$\begin{cases} \sigma_1 d\widetilde{B}^1 - \sigma_2 d\widetilde{B}^2 = (\sigma_1 - \sigma_2\alpha)d\widetilde{W}^1 - \sigma_2\sqrt{1-\alpha^2}d\widetilde{W}^2 \\ \sigma_2 d\widetilde{B}^2 - \sigma_1 d\widetilde{B}^1 = \sigma_2\sqrt{1-\alpha^2}d\widetilde{W}^2 - (\sigma_1 - \sigma_2\alpha)d\widetilde{W}^1 \end{cases}$$

and since $\widetilde{W}^1$ and $\widetilde{W}^2$ are independent Brownian motions, we may argue analogously to (b) that X^1 and X^2 are true $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingales provided that (5) holds.

Alternatively, if we can argue that the (strictly non-negative) local martingale X^i is bounded from below and hence a $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-supermartingale, and that X^i has a constant mean equal to the initial value, we can conclude that X^i is a $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingale.