

Mathematical Foundations for Finance

Exercise sheet 13

Please upload your solutions until Wednesday, 22/12/2021, 12:00 using the link on the course website.

Exercise 13.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. Let $M = (M_t)_{t \geq 0}$ be a local (P, \mathbb{F}) -martingale and $W = (W_t)_{t \geq 0}$ a (P, \mathbb{F}) -Brownian motion.

(a) Let $H = (H_t)_{t \geq 0}$ be in $L^2(M)$. Compute $E \left[\int_0^T H_s dM_s \right]$ and $\text{Var} \left[\int_0^T H_s dM_s \right]$. How do the expressions look for $M := W$?

(b) Let $H_s := \exp(-4s)$. Show that $\int_0^T H_s dW_s$ is in fact normally distributed. What are the mean and the variance of this normal distribution? How would the result change if $H : \mathbb{R} \rightarrow \mathbb{R}$ were an arbitrary (deterministic) continuous function?

Hint 1: Use the dominated convergence theorem for stochastic integrals from page 94 in the lecture notes.

Hint 2: If $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$, $X_n \rightarrow X$ in probability, $\mu_n \rightarrow \mu$ and $\sigma_n^2 \rightarrow \sigma^2 > 0$, then $X \sim \mathcal{N}(\mu, \sigma^2)$.

(c) By coming up with a counterexample, show that the normality of $\int_0^T H_s dW_s$ from (b) does not hold for an arbitrary continuous $H \in L^2(W)$.

Exercise 13.2 Let $T > 0$ denote a fixed time horizon and $W = (W_t)_{t \in [0, T]}$ a Brownian motion on some probability space (Ω, \mathcal{F}, P) . Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ be the filtration generated by W and augmented by the P -nullsets in $\sigma(W_s; s \leq T)$. Consider the Black-Scholes model, where the undiscounted bank account price process $\tilde{S}^0 = (\tilde{S}_t^0)_{t \in [0, T]}$ and the undiscounted stock price process $\tilde{S}^1 = (\tilde{S}_t^1)_{t \in [0, T]}$ are given by

$$d\tilde{S}_t^0 = \tilde{S}_t^0 r dt \quad \text{and} \quad d\tilde{S}_t^1 = \tilde{S}_t^1 (\mu dt + \sigma dW_t), \quad (1)$$

where $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\tilde{S}_0^0 = 1$ and $\tilde{S}_0^1 > 0$ are deterministic.

(a) Prove using Itô's formula that the discounted stock price process $S^1 = \tilde{S}^1 / \tilde{S}^0$ solves

$$dS_t^1 = S_t^1 ((\mu - r)dt + \sigma dW_t). \quad (2)$$

(b) Prove using Itô's formula that

$$S^1 = \left(S_0^1 \exp \left(\sigma W_t + \left(\mu - r - \frac{1}{2} \sigma^2 \right) t \right) \right)_{t \in [0, T]},$$

i.e. show that the process $(S_0^1 \exp(\sigma W_t + (\mu - r - \frac{1}{2} \sigma^2) t))_{t \in [0, T]}$ solves (2).

(c) Let $L^\lambda := -\lambda W$ and $Z^\lambda := \mathcal{E}(L^\lambda)$. Prove that the process $W^\lambda := (W_t + \lambda t)_{t \in [0, T]}$ is a Brownian motion under the measure Q_λ given by $\frac{dQ_\lambda}{dP} := Z_T^\lambda$.

(d) Prove that for the right choice of λ , the discounted stock price process S^1 is a Q_λ -martingale.
Hint: Rewrite $\sigma W_t + (\mu - r - \frac{1}{2} \sigma^2) t$ as function of $W_t^\lambda, t, \sigma, \mu$, and r .

Exercise 13.3 Let $T > 0$ denote a fixed time horizon and let $W = (W_t)_{t \in [0, T]}$ be a Brownian motion on some probability space (Ω, \mathcal{F}, P) . Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ be the filtration generated by W and augmented by the P -nullsets in $\sigma(W_s; 0 \leq s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\tilde{S}^0 = (\tilde{S}_t^0)_{t \in [0, T]}$ and the undiscounted stock price process $\tilde{S}^1 = (\tilde{S}_t^1)_{t \in [0, T]}$ are given by

$$\frac{d\tilde{S}_t^0}{\tilde{S}_t^0} = rdt \quad \text{and} \quad \frac{d\tilde{S}_t^1}{\tilde{S}_t^1} = \mu dt + \sigma dW_t,$$

with $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\tilde{S}_0^0 = 1$ and $\tilde{S}_0^1 > 0$ deterministic. Using the notation of the previous exercise, denote $Q^* := Q_{\lambda^*}$, where λ^* is the unique value of λ making Q_λ an equivalent martingale measure for $S^1 := \tilde{S}^1 / \tilde{S}^0$.

Hint: If you did not find λ^ in Exercise 13.2 (d), you can use that $\lambda^* = \frac{\mu - r}{\sigma}$.*

- (a) Hedge the *square option*, i.e., find a self-financing strategy $\varphi \hat{=} (V_0, \vartheta)$ such that

$$V_0 + \int_0^T \vartheta_u dS_u^1 = \frac{(\tilde{S}_T^1)^2}{\tilde{S}_T^0}.$$

Hint: Look for a representation result under Q^ , not under P .*

- (b) Hedge the *inverted option*, i.e., find a self-financing strategy $\varphi \hat{=} (\bar{V}_0, \bar{\vartheta})$ such that

$$\bar{V}_0 + \int_0^T \bar{\vartheta}_u dS_u^1 = \frac{1}{\tilde{S}_T^0 \tilde{S}_T^1}.$$