Mathematical Foundations for Finance

Exercise sheet 13

Please upload your solutions until Wednesday, 22/12/2021, 12:00 using the link on the course website.

Exercise 13.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Let $M = (M_t)_{t\geq 0}$ be a local $(P, \mathbb{F})$-martingale and $W = (W_t)_{t\geq 0}$ a $(P, \mathbb{F})$-Brownian motion.

(a) Let $H = (H_t)_{t\geq 0}$ be in $L^2(M)$. Compute $E\left[\int_0^T H_s dM_s\right]$ and $\mathrm{Var}\left[\int_0^T H_s dM_s\right]$. How do the expressions look for $M := W$?

(b) Let $H_s := \exp(-4s)$. Show that $\int_0^T H_s dW_s$ is in fact normally distributed. What are the mean and the variance of this normal distribution? How would the result change if $H : \mathbb{R} \to \mathbb{R}$ were an arbitrary (deterministic) continuous function?
Hint 1: Use the dominated convergence theorem for stochastic integrals from page 94 in the lecture notes.
Hint 2: If $X_n \sim \mathcal{N}\left(\mu_n, \sigma_n^2\right)$, $X_n \to X$ in probability, $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2 > 0$, then $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

(c) By coming up with a counterexample, show that the normality of $\int_0^T H_s dW_s$ from (b) does not hold for an arbitrary continuous $H \in L^2(W)$.

Solution 13.1

(a) Since $H \in L^2(M)$, we know that $(H \bullet M) \in \mathcal{M}_0^2$, i.e. $(H \bullet M)$ is a square-integrable RCLL $(P, \mathbb{F})$-martingale null at 0 with $\sup_{t\geq 0} E\left[(H \bullet M)_T^2\right] < \infty$. In particular, both $E\left[(H \bullet M)_T\right]$ and $\mathrm{Var}\left[(H \bullet M)_T\right]$ are finite for all $T \geq 0$. We must therefore have that

$$E\left[\int_0^T H_s dM_s\right] = E\left[E\left[\int_0^T H_s dM_s \,\middle|\, \mathcal{F}_0\right]\right] = E\left[(H \bullet M)_0\right] = 0.$$

For the variance, we compute

$$\mathrm{Var}\left[\int_0^T H_s dM_s\right] = E\left[\left(\int_0^T H_s dM_s\right)^2\right] = E\left[\int_0^T H_s^2 d[M]_s\right],$$

where the first equality uses that $E\left[(H \bullet M)_T\right] = 0$ and the second exploits the isometry property of the stochastic integral with respect to a local martingale.

In the particular case of Brownian motion, i.e. when $M = W$, no further simplification is needed for $E\left[(H \bullet W)_T\right]$. As for $\mathrm{Var}\left[(H \bullet W)_T\right]$, we can compute

$$E\left[\int_0^T H_s^2 d[W]_s\right] = E\left[\int_0^T H_s^2 ds\right] = \int_0^T E\left[H_s^2\right] ds,$$

where the last equality uses Fubini's theorem.

(b) Let us first discuss the hint. It is known that convergence in probability implies convergence in distribution, where sequence (X_n) of random variables converges in distribution (weakly) to a random variable X if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for every $x \in \mathbb{R}$ at which F is continuous. Here F_n is the distribution function of X_n and F the distribution function of X. But using the standard notation for the distribution function of $\mathcal{N}(0,1)$, we have that

$$F_n(x) = \Phi\left(\frac{x-\mu_n}{\sigma_n}\right) \quad \text{and} \quad F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

and since Φ is a continuous function and $\sigma_n^2 \to \sigma > 0$, we have that

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \Phi\left(\frac{x-\mu_n}{\sigma_n}\right) = \Phi\left(\frac{x-\lim_{n\to\infty}\mu_n}{\lim_{n\to\infty}\sigma_n}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right) = F(x)$$

for all $x \in \mathbb{R}$. We can thus conclude that (X_n) converges in distribution to a random variable $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

Now let $(\Pi_n)_{n\in\mathbb{N}}$ be a sequence of refining partitions of $[0,\infty)$ with $|\Pi_n| \to 0$ and define

$$H_s^n := \sum_{t_i\in\Pi_n} H_{t_i} \mathbb{1}_{(t_i,t_{i+1}]}(s) = \sum_{t_i\in\Pi_n} e^{-4t_i} \mathbb{1}_{(t_i,t_{i+1}]}(s). \tag{1}$$

Since we have that $H_s^n \to H_s$ pointwise for all $s \in \mathbb{R}_+$, then also $(H_s^n - H_s) \to 0$ pointwise for all $s \in \mathbb{R}_+$. But since all H^n and H are independent of $\omega \in \Omega$, we also have the pointwise convergence on $\overline{\Omega} := \Omega \times \mathbb{R}_+$. At the same time, we have for all $s \in \mathbb{R}_+$ that

$$\left|e^{-4s}\right| \leq 1 \implies |H_s^n - H_s| \leq 2, \tag{2}$$

so the dominated convergence theorem for stochastic integral from page 94 in the lecture notes says that $(H^n - H) \bullet W \to 0$ uniformly on compacts in probability (ucp), i.e.

$$\lim_{n\to\infty} P\left[\sup_{0\leq s\leq T} \left|\left((H^n-H)\bullet W\right)_s\right| > \epsilon\right] = 0 \quad \text{for all } \epsilon > 0 \text{ and for all } t \geq 0.$$

But the above implies that

$$\lim_{n\to\infty} P\left[\left|\left((H^n-H)\bullet W\right)_T\right| > \epsilon\right] = \lim_{n\to\infty} P\left[\left|(H^n\bullet W)_T - (H\bullet W)_T\right| > \epsilon\right] = 0 \quad \text{for all } \epsilon > 0,$$

so $(H^n \bullet W)_T \to (H\bullet W)_T$ in probability for all $T \geq 0$. Moreover,

$$(H^n\bullet W)_T = \sum_{t_i\in\Pi_n} H_{t_i}(W_{t_{i+1}\wedge T} - W_{t_i\wedge T}) \sim \mathcal{N}\left(m_n, s_n^2\right)$$

with

$$m_n = 0 \quad \text{and} \quad s_n^2 = \sum_{t_i\in\Pi_n} H_{t_i}^2 (t_{i+1}\wedge T - t_i \wedge T),$$

since Π_n is finite for all $n \in \mathbb{N}$ and the increments $W_{t_{i+1}\wedge T} - W_{t_i\wedge T}$ are independent and normally distributed with $\mathcal{N}(0, t_{i+1}\wedge T - t_i\wedge T)$. From the above, we see that

$$m_n \to 0 \quad \text{and} \quad s_n^2 \to \int_0^T H_s^2 ds = \int_0^T e^{-8s} ds = \frac{1}{8}\left(1 - e^{-8T}\right).$$

Hint 2 also gives us that the limiting distribution is additionally normal, so

$$\int_0^T e^{-4s} dW_s \sim \mathcal{N}\left(0, \frac{1}{8}\left(1-e^{-8T}\right)\right)$$

The only difference for an arbitrary continuous $H : \mathbb{R} \to \mathbb{R}$ is that the bound in (2) does not work. However for a fixed $T \geq 0$ we have that $H^n_s \mathbb{1}_{[0,T]}(s) \to H_s \mathbb{1}_{[0,T]}(s)$ pointwise for all $s \in \mathbb{R}_+$ so $|H^n_s \mathbb{1}_{[0,T]}(s) - H_s \mathbb{1}_{[0,T]}(s)| \to 0$ pointwise for all $s \in \mathbb{R}_+$. Since both $H^n \mathbb{1}_{[0,T]}$ and $H\mathbb{1}_{[0,T]}$ are bounded, so is $|H^n \mathbb{1}_{[0,T]} - H\mathbb{1}_{[0,T]}|$. Additionally, $H^n \mathbb{1}_{[0,T]}$ and $H\mathbb{1}_{[0,T]}$ are also left-continuous and therefore predictable, so the dominated convergence theorem is indeed applicable and we can proceed as before. We obtain that

$$\int_0^T H^n_s \mathbb{1}_{[0,T]}(s) dW_s \to \int_0^T H_s \mathbb{1}_{[0,T]}(s) dW_s = \int_0^T H_s dW_s \quad \text{in probability.}$$

(c) We have seen in the lecture (see for instance the example on page 102 of the lecture notes) that

$$\int_0^T W_s dW_s = \frac{1}{2} W_T^2 - \frac{1}{2} T.$$

Since W_T^2 only takes positive values, it is clear that $\frac{1}{2} W_T^2 - \frac{1}{2} T$ is bounded from below for every $T \geq 0$, so it cannot be normally distributed. More specifically,

$$\frac{1}{2} W_T^2 - \frac{1}{2} T \overset{d}{=} \frac{1}{2} (\sqrt{(T)} W_1)^2 - \frac{1}{2} T,$$

and since $W_1 \sim \mathcal{N}(0,1)$, we have that $W_1^2 \sim \chi_1^2$, so the distribution of $\int_0^T W_s dW_s$ is just a simple affine transformation of a chi-squared distribution with one degree of freedom.

Exercise 13.2 Let $T > 0$ denote a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ are given by

$$d\widetilde{S}^0_t = \widetilde{S}^0_t r dt \quad \text{and} \quad d\widetilde{S}^1_t = \widetilde{S}^1_t \left(\mu dt + \sigma dW_t\right), \tag{3}$$

where $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}^0_0 = 1$ and $\widetilde{S}^1_0 > 0$ are deterministic.

(a) Prove using Itô's formula that the discounted stock price process $S^1 = \widetilde{S}^1 / \widetilde{S}^0$ solves

$$dS^1_t = S^1_t \left((\mu - r)dt + \sigma dW_t\right). \tag{4}$$

(b) Prove using Itô's formula that

$$S^1 = \left(S^1_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2 \right) t \right) \right)_{t\in[0,T]},$$

i.e. show that the process $\left(S^1_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t\right)\right)_{t\in[0,T]}$ solves (4).

(c) Let $L^\lambda := -\lambda W$ and $Z^\lambda := \mathcal{E}(L^\lambda)$. Prove that the process $W^\lambda := (W_t + \lambda t)_{t\in[0,T]}$ is a Brownian motion under the measure Q_λ given by $\frac{dQ_\lambda}{dP} := Z^\lambda_T$.

(d) Prove that for the right choice of λ, the discounted stock price process S^1 is a Q_λ-martingale. *Hint: Rewrite* $\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t$ *as function of* $W^\lambda_t, t, \sigma, \mu,$ *and* r.

Solution 13.2

(a) Using that $\widetilde{S}^0 > 0$ P-a.s., we can apply Itô's formula to the C^2-function $f : \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ given by $f(x, y) := x/y$. Computing the required derivatives, we obtain

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{y}, \quad \frac{\partial f}{\partial y}(x,y) = -\frac{x}{y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(x,y) = 0,$$

and, moreover, since $\widetilde{S}^0$ is of finite variation,

$$\langle \widetilde{S}^1, \widetilde{S}^0 \rangle = 0 \quad \text{and} \quad \langle \widetilde{S}^0, \widetilde{S}^0 \rangle = 0.$$

By Itô's formula, we obtain

$$\begin{aligned} S_t^1 = \frac{\widetilde{S}_t^1}{\widetilde{S}_t^0} &= S_0^1 + \int_0^t \frac{1}{\widetilde{S}_s^0} d\widetilde{S}_s^1 + \int_0^t \frac{-\widetilde{S}_s^1}{(\widetilde{S}_s^0)^2} d\widetilde{S}_s^0 + 0 \\ &= S_0^1 + \int_0^t \frac{1}{\widetilde{S}_s^0} \left(\widetilde{S}_s^1 \left(\mu ds + \sigma dW_s \right) \right) + \int_0^t \frac{-\widetilde{S}_s^1}{(\widetilde{S}_s^0)^2} \left(\widetilde{S}_s^0 r ds \right) \\ &= S_0^1 + \int_0^t S_s^1 \left((\mu - r) ds + \sigma dW_s \right), \end{aligned}$$

or, written equivalently in the differential notation,

$$dS_t^1 = S_t^1 \left((\mu - r) dt + \sigma dW_t \right).$$

(b) Consider the C^2-function $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ given by $f(w,t) := S_0^1 \exp\left(\sigma w + (\mu - r - \sigma^2/2)t \right)$. Computing the different derivatives, we obtain

$$\frac{\partial f}{\partial w}(w,t) = \sigma f(w,t), \quad \frac{\partial f}{\partial t}(w,t) = (\mu - r - \sigma^2/2) f(w,t), \quad \text{and} \quad \frac{\partial^2 f}{\partial w^2}(w,t) = \sigma^2 f(w,t),$$

and moreover, since $(t)_{t \in [0,T]}$ is of finite variation,

$$\langle W \rangle_t = t, \quad \langle t, W \rangle = \langle W, t \rangle = 0, \quad \text{and} \quad \langle t \rangle = 0.$$

By Itô's formula, we can then compute

$$\begin{aligned} f(W_t, t) &= S_0^1 + \int_0^t \sigma f(W_s, s) dW_s + \int_0^t \left(\mu - r - \frac{\sigma^2}{2} \right) f(W_s, s) ds + \frac{1}{2} \int_0^t \sigma^2 f(W_s, s) ds \\ &= S_0^1 + \int_0^t f(W_s, s) \left(\sigma dW_s + (\mu - r) \, ds \right), \end{aligned}$$

and thus conclude that $f(W_t, t) = S_0^1 \exp\left(\sigma W_t + (\mu - r - \sigma^2/2)t \right)$ solves the SDE.

(c) Since $\lambda t = \int_0^t \lambda ds$, we can write $W_t^\lambda = W_t + \int_0^t \lambda ds$. Using that $L_t = -\lambda W_t = \int_0^t -\lambda dW_s$, we can then directly conclude by Girsanov's theorem (Theorem VI.2.3 in the lecture notes) that W^λ is a Q_λ-Brownian motion.

(d) Note that we can write

$$\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2 \right) t = \sigma(W_t^\lambda - \lambda t) + \left(\mu - r - \frac{1}{2}\sigma^2 \right) t = \sigma W_t^\lambda + \left(\mu - r - \frac{1}{2}\sigma^2 - \sigma\lambda \right) t.$$

Since by (c) W^λ is a Brownian motion under Q_λ, we can deduce from Proposition IV.2.2 in the lecture notes that the process

$$\left(S_0^1 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2 \right) t \right) \right)_{t \in [0,T]} \tag{5}$$

is a Q_λ-martingale if and only if $\mu - r - \frac{1}{2}\sigma^2 - \sigma\lambda = -\frac{1}{2}\sigma^2$. Solving this equation, we can conclude by (b), since S^1 coincides with the process in (5), that S^1 is a Q^λ martingale if and only if $\lambda = \frac{\mu - r}{\sigma}$.

Exercise 13.3 Let $T > 0$ denote a fixed time horizon and let $W = (W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \le s \le T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ are given by

$$\frac{d\widetilde{S}^0_t}{\widetilde{S}^0_t} = r dt \quad \text{and} \quad \frac{d\widetilde{S}^1_t}{\widetilde{S}^1_t} = \mu dt + \sigma dW_t,$$

with $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}^0_0 = 1$ and $\widetilde{S}^1_0 > 0$ deterministic. Using the notation of the previous exercise, denote $Q^* := Q_{\lambda^*}$, where λ^* is the unique value of λ making Q_λ an equivalent martingale measure for $S^1 := \widetilde{S}^1/\widetilde{S}^0$.
Hint: If you did not find λ^* in Exercise 13.2 (d), you can use that $\lambda^* = \frac{\mu-r}{\sigma}$.

(a) Hedge the *square option*, i.e., find a self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ such that

$$V_0 + \int_0^T \vartheta_u dS^1_u = \frac{(\widetilde{S}^1_T)^2}{\widetilde{S}^0_T}.$$

Hint: Look for a representation result under Q^*, not under P.

(b) Hedge the *inverted option*, i.e., find a self-financing strategy $\varphi \mathrel{\widehat{=}} (\overline{V}_0, \overline{\vartheta})$ such that

$$\overline{V}_0 + \int_0^T \overline{\vartheta}_u dS^1_u = \frac{1}{\widetilde{S}^0_T \widetilde{S}^1_T}.$$

Solution 13.3 Set $\lambda^* := \frac{\mu-r}{\sigma}$. We know from the previous exercise that $W^*_t := W_t + \lambda^* t$, $t \in [0,T]$ is a Brownian motion under the equivalent martingale measure Q^*. Moreover, the discounted stock price process $S^1 = \frac{\widetilde{S}^1}{\widetilde{S}^0}$ is a Q^*-martingale and is explicitly given by

$$S_t = S^1_0 e^{\sigma W^*_t - \frac{1}{2}\sigma^2 t}.$$

The *discounted* arbitrage-free value at time t of any *discounted* payoff $H \in L^1_+(\mathcal{F}_T, Q^*)$ is given by

$$V^*_t = E_{Q^*}\left[H \,|\, \mathcal{F}_t\right].$$

(a) We use the discussion from page 114 of the lecture notes to conclude that $V^*_t = E_{Q^*}\left[H \,|\, \mathcal{F}_t\right]$ may be represented as a stochastic integral of the form

$$V^*_t = E_{Q^*}\left[H\right] + \int_0^t \varphi_s dS^1_s, \quad 0 \le t \le T.$$

In this case, we compute

$$\begin{aligned} V^*_t &= e^{-rT} E_{Q^*}\left[\left(\widetilde{S}^1_T\right)^2 \,\middle|\, \mathcal{F}_t\right] = e^{-rT} e^{2Tr} E_{Q^*}\left[\left(S^1_T\right)^2 \,\middle|\, \mathcal{F}_t\right] \\ &= e^{rT}\left(S^1_t\right)^2 E_{Q^*}\left[e^{2\sigma(W^*_T - W^*_t) - \sigma^2(T-t)} \,\middle|\, \mathcal{F}_t\right] \\ &= e^{(r+\sigma^2)T - \sigma^2 t}\left(S^1_t\right)^2 =: v\big(t, S^1_t\big). \end{aligned} \tag{6}$$

We apply Itô's formula to v and obtain

$$v(t, S^1_t) = v(0, S^1_0) + \int_0^t \frac{\partial}{\partial x} v(t, S^1_t) dS^1_t + \text{ continuous FV process.}$$

Since the left-hand side and the stochastic integral on the right-hand side are local $(Q^*, \mathbb{F})$-martingales, the "continuous FV process" is a local $(Q^*, \mathbb{F})$-martingale as well and since it apparently is null at 0, it must be identically equal to 0 (see the second hint from Exercise 11.1 (d)). We thus immediately obtain that

$$\vartheta_t = \frac{\partial}{\partial x} v(t, S_t^1) = 2e^{(r+\sigma^2)T - \sigma^2 t} S_t^1 = 2e^{(r+\sigma^2)T + (r-\sigma^2)t} \widetilde{S}_t^1.$$

For $v(0, S_0^1)$, we have that $v(0, S_0^1) = e^{(r+\sigma^2)T}(S_0^1)^2$.

(b) We proceed analogously to (a). Here, we obtain

$$\begin{aligned}
\overline{V}_t^* &= e^{-rT} E_{Q^*}\left[\frac{1}{\widetilde{S}_T^1} \,\middle|\, \mathcal{F}_t\right] = e^{-2rT} \frac{1}{S_t^1} E_{Q^*}\left[\frac{S_t^1}{S_T^1} \,\middle|\, \mathcal{F}_t\right] \\
&= e^{-2rT} \frac{1}{S_t^1} E_{Q^*}\left[e^{-\sigma(W_T^* - W_t^*) + \frac{1}{2}\sigma^2(T-t)} \,\middle|\, \mathcal{F}_t\right] \\
&= e^{(\sigma^2 - 2r)T - \sigma^2 t} \frac{1}{S_t^1} \\
&=: \overline{v}(t, S_t^1).
\end{aligned}$$

We conclude again in the same way as in (a) that

$$\begin{aligned}
\overline{\vartheta}_t &= \frac{\partial}{\partial x} \overline{v}(t, S_t^1) = -e^{(\sigma^2 - 2r)T - \sigma^2 t} \frac{1}{(S_t^1)^2} = -e^{(\sigma^2 - 2r)(T-t)} \frac{1}{(\widetilde{S}_t^1)^2}, \\
\overline{V}_0^* &= v(0, S_0^1) = e^{(\sigma^2 - 2r)T} \frac{1}{S_0^1} = e^{(\sigma^2 - 2r)T} \frac{1}{\widetilde{S}_0^1}.
\end{aligned}$$