Mathematical Foundations for Finance

Exercise sheet 14

This exercise sheet will not be corrected. Please do not hand in.

Exercise 14.1 We will now use the techniques that we have developed in the course in a slightly different setting. Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ on a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let W^1 and W^2 be two $(P, \mathbb{F})$-Brownian motions with (constant) correlation $\rho \in (-1,1)$[1] and let the dynamics of $\widetilde{S}^0$ and $\widetilde{S}^1$ be described by the SDEs

$$\begin{aligned} d\widetilde{S}^0_t &= \widetilde{S}^0_t r_t dt, \\ dr_t &= \theta(\alpha - r_t)dt + \eta dW^1_t, \\ d\widetilde{S}^1_t &= \widetilde{S}^1_t \left(r_t dt + \sigma dW^2_t\right), \end{aligned}$$

where $\sigma, \eta > 0$ and $\theta, \alpha \in \mathbb{R}$ as well as $\widetilde{S}^0_0 = 1$, $\widetilde{S}^1_0 > 0$ and $r_0 \in \mathbb{R}$ are all constant. This is the Black–Scholes model with stochastic interest rate.

(a) By applying Itô's formula to some function $f \in C^2$ and the continuous semimartingale $\widetilde{S}^0$, show that the solution to the first SDE is given by

$$\widetilde{S}^0_t = \exp\left(\int_0^t r_s ds\right).$$

(b) By applying Itô's formula to the function $f(x,t) = xe^{\theta t}$ and the continuous semimartingale $(r_t, t)_{t\geq 0}$, show that the solution to the second SDE is given by

$$r_t = r_0 e^{-\theta t} + \alpha(1 - e^{-\theta t}) + \eta e^{-\theta t} \int_0^t e^{\theta s} dW_s.$$

The solution to this SDE is called *Ornstein–Uhlenbeck process* and is an important process when it comes to interest rate modeling.

(c) Show that the discounted price processes $S^0 := \widetilde{S}^0/\widetilde{S}^0$ and $S^1 := \widetilde{S}^1/\widetilde{S}^0$ are $(P, \mathbb{F})$-martingales, i.e. the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free and we can use P as our pricing measure.

Solution 14.1

(a) Due to the similarity with the ordinary differential equation $\frac{y'}{y} = g \iff \log(y)' = g$, whose solution is given by $y(t) = C\exp\left(\int g(t)dt\right)$, one might try to apply Itô's formula to the function $f(x) = \log(x)$ and the positive continuous semimartingale $\widetilde{S}^0$. This yields

$$\begin{aligned} \log(\widetilde{S}^0_t) &= \log(\widetilde{S}^0_0) + \int_0^t \frac{1}{\widetilde{S}^0_s} d\widetilde{S}^0_s - \frac{1}{2}\int_0^t \frac{1}{(\widetilde{S}^0_t)^2} d[\widetilde{S}^0]_s \\ &= \int_0^t \frac{1}{\widetilde{S}^0_s}\widetilde{S}^0_s r_s ds = \int_0^t r_s ds, \end{aligned}$$

[1] We say that two Brownian motions W^1 and W^2 on the same probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ are *correlated with correlation* $\rho \in [-1,1]$ if for $s \leq t$, the increments $W^1_t - W^1_s$ and $W^2_t - W^2_s$ are independent of $\mathcal{F}_s$ and jointly normally distributed with $\mathcal{N}(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} t-s & \rho(t-s) \\ \rho(t-s) & t-s \end{pmatrix}.$$

where we have used that $\widetilde{S}^0$ is of finite variation and therefore

$$[\widetilde{S}^0]_t = \left[\int \widetilde{S}^0 r ds\right]_t = \int_0^s (\widetilde{S}^0_s)^2 r_s^2 d[s]_s = 0,$$

so indeed

$$\widetilde{S}^0_t = \exp\left(\int_0^t r_s ds\right).$$

(b) Since we have that $[t] = 0$ and $[W, t] = 0$, Itô's formula gives

$$f(r_t, t) = f(r_0, 0) + \int_0^t \frac{\partial f}{\partial x}(r_s, s) dr_s + \int_0^t \frac{\partial f}{\partial t}(r_s, s) ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(r_s, s) d[r]_s.$$

The required derivatives read

$$\frac{\partial f}{\partial x}(x,t) = e^{\theta t}, \quad \frac{\partial f}{\partial t}(x,t) = \theta x e^{\theta t}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(x,t) = 0.$$

Using the above and the fact that $d[r]_t = \eta^2 d[W^1]_t = \eta^2 dt$, we obtain

$$\begin{aligned} r_t e^{\theta t} &= r_0 + \int_0^t e^{\theta s}\theta\alpha ds - \int_0^t e^{\theta s}\theta r_s ds + \int_0^t e^{\theta s}\eta dW_s^1 + \int_0^t e^{\theta s}\theta r_s ds \\ &= r_0 + \theta\alpha \int_0^t e^{\theta s} ds + \eta\int_0^t e^{\theta s} dW_s^1 \\ &= r_0 + \alpha(e^{\theta t} - 1) + \eta\int_0^t e^{\theta s} dW_s^1. \end{aligned}$$

Multiplying both sides of the equation by $e^{-\theta t}$ gives the desired result.

(c) Since $S^0 \equiv 1$, it is clearly a $(P, \mathbb{F})$-martingale. As for $S^1 = \widetilde{S}^1/\widetilde{S}^0$, we do this by applying Itô's formula to the C^2 function $f(x,y) = \frac{x}{y}$ and the semimartingale $(\widetilde{S}^1_t, \widetilde{S}^0_t)_{t\geq 0}$. Since we have seen in (a) that $\widetilde{S}^0$ is of finite variation and $[\widetilde{S}^0] = 0$, we also have that $[\widetilde{S}^1, \widetilde{S}^0] = [\widetilde{S}^0, \widetilde{S}^1] = 0$. Itô's formula therefore gives

$$S^1_t = S^1_0 + \int_0^t \frac{\partial f}{\partial x}(\widetilde{S}^1_s, \widetilde{S}^0_s) d\widetilde{S}^1_s + \int_0^t \frac{\partial f}{\partial y}(\widetilde{S}^1_s, \widetilde{S}^0_s) d\widetilde{S}^0_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(\widetilde{S}^1_s, \widetilde{S}^0_s) d[\widetilde{S}^1]_s.$$

A direct computation of the required derivatives yields

$$\frac{\partial}{\partial x} f(x,y) = \frac{1}{y}, \quad \frac{\partial}{\partial y} f(x,y) = -\frac{x}{y^2}, \quad \frac{\partial^2}{\partial x^2} f(x,y) = 0, \tag{1}$$

giving us in turn that

$$S^1_t = S^1_0 + \int_0^t S^1_s r_s ds + \sigma\int_0^t S^1_s dW^2_s - \int_0^t S^1_s r_s ds = S^1_0 + \sigma\int_0^t S^1_s dW^2_s. \tag{2}$$

Note that we can rewrite (2) in the differential form as $dS^1_t = \sigma S^1_t dW^2_t$. In order to show that S^1 is in fact a true $(P, \mathbb{F})$-martingale, we apply Itô's formula to the C^2 function $f(x) = \log(x)$ and the local $(P, \mathbb{F})$-martingale (thus a $(P, \mathbb{F})$-semimartingale) S^1. Similarly to (a), this yields

$$\begin{aligned} \log(S^1_t) &= \log(S^1_0) + \int_0^t \sigma \frac{1}{S^1_s} S^1_s dW^2_s - \frac{1}{2}\int_0^t \sigma^2 \frac{1}{(S^1_s)^2}(S^1_s)^2 ds \\ &= \log(S^1_0) + \sigma W^2_t - \frac{1}{2}\sigma^2 t, \end{aligned}$$

or, equivalently, $S_t^1 = S_0^1 \exp\left(\sigma W_t^2 - \frac{1}{2}\sigma^2 t\right)$. Alternatively, $dS_t^1 = S_t^1 \sigma dW_t^2$ gives directly that

$$S_t^1 = S_0^1 \mathcal{E}(\sigma W^2)_t = S_0^1 \exp\left(\sigma W_t^2 - \frac{1}{2}\sigma^2 t\right).$$

But as we have seen in Proposition IV.2.2 in the lecture notes, the exponential term is a $(P, \mathbb{F})$-martingale, and since S_0^1 is just a constant, S^1 is also a $(P, \mathbb{F})$-martingale.

Exercise 14.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a Brownian motion $W = (W_t)_{t\in[0,T]}$. Let $\mathbb{F} := \mathbb{F}^W$ be the (augmented) filtration generated by W. Consider the discounted price process $S = (S_t)_{t\in[0,T]}$ with dynamics

$$dS_t = \sigma(t, S_t) dW_t, \quad S_0 > 0,$$

where $\sigma : [0, T] \times \mathbb{R} \to (0, \infty)$ is a continuous and bounded function. One can show that S is well defined and P is the unique EMM for S. Let $h : \mathbb{R} \to [0, \infty)$ be a fixed continuous and bounded function. We consider the partial differential equation (PDE)

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2} v(t,x) = 0, & x \in (0,T) \times \mathbb{R}, \\ v(T,x) = h(x), & x \in \mathbb{R}. \end{cases} \tag{3}$$

Suppose that there exists a $C^{1,2}$ solution $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ of (3) with the additional property that

$$\left|\sigma(t,x)\frac{\partial}{\partial x} v(t,x)\right| \le C(1 + |x|), \quad (t,x) \in [0,T] \times \mathbb{R},$$

for some constant $C > 0$. Show that $V_t^* := v(t, S_t)$, $t \in [0, T]$, is the price at time t of the discounted European contingent claim $h(S_T)$.

Solution 14.2 By Itô's formula, we have

$$\begin{aligned} V_t^* = v(t, S_t) &= v(0, S_0) + \int_0^t \frac{\partial}{\partial x} v(t, S_t) dS_t + \int_0^t \left(\frac{\partial}{\partial t} v(t, S_t) + \frac{1}{2}\sigma^2(t, S_t)\frac{\partial^2}{\partial x^2} v(t, S_t)\right) dt \\ &= v(0, S_0) + \int_0^t \frac{\partial}{\partial x} v(t, S_t) dS_t, \end{aligned}$$

where the "dt" integral disappears by the assumption that v solves the given PDE. We now claim that the stochastic integral $\int \frac{\partial}{\partial x} v(t, S_t) dS_t$ is in fact a square-integrable $(P, \mathbb{F})$-martingale. To that end, we recall that since σ is a bounded function and $\sup_{0\le t\le T} E\left[W_t^2\right] \le T$, i.e. $W^T \in \mathcal{M}_0^2$, S^T is a $(P, \mathbb{F})$-martingale in $\mathcal{M}_0^2$ by Proposition V.1.3 in the lecture notes. Hence $\sup_{0\le u\le T} |S_u| \in L^2(P)$. Moreover,

$$\begin{aligned} \left\|\frac{\partial}{\partial x} v(t, S_t)\right\|_{L^2(S)}^2 &= E\left[\left(\int_0^T \frac{\partial}{\partial x} v(t, S_t) dS_t\right)^2\right] = E\left[\int_0^T \left(\frac{\partial}{\partial x} v(t, S_t)\right)^2 d[S]_t\right] \\ &= E\left[\int_0^T \left(\sigma(t, S_t)\frac{\partial}{\partial x} v(t, S_t)\right)^2 dt\right] \\ &\le E\left[C^2 T \left(1 + \sup_{0\le t\le T} |S_t|\right)^2\right] < \infty, \end{aligned}$$

due to the assumption that

$$\left|\sigma(t,x)\frac{\partial}{\partial x} v(t,x)\right| \le C(1 + |x|), \quad (t,x) \in [0,T] \times \mathbb{R},$$

and the fact that $\sup_{0\le u\le T}|S_u| \in L^2(P)$. Since $\frac{\partial}{\partial x}v(t,S)$ is continuous and $\mathbb{F}$-adapted, therefore predictable, it means that $\frac{\partial}{\partial x}v(t,S) \in L^2(S)$ and the stochastic integral $\int \frac{\partial}{\partial x}v(t,S_t)dS_t$ is in fact a $(P,\mathbb{F})$-martingale on $[0,T]$. We conclude that V^* is a true $(P,\mathbb{F})$-martingale with $V_T^* = h(S_T)$. So

$$v(t,S_t) = V_t^* = E_P\left[v(T,S_T)\,|\,\mathcal{F}_t\right] = E_P\left[h(S_T)\,|\,\mathcal{F}_t\right]$$

is the price at time t of the claim $h(S_T)$.

Exercise 14.3 The exam will contain two parts. Part I is a series of 8 multiple choice questions while part II consists of 4 open questions (2 discrete time and 2 continuous time problems). The goal of this exercise is to familiarize yourself with the type of multiple choice questions that were asked in part I of previous exams. For each of the following subquestions, there is **exactly one** correct answer. Please note that in part I of the exam, you do not need to explain your choice and for each correct answer you get 1 point, for each wrong answer you get -0.5 point, and for no answer you get 0 points. You get at least 0 points for the whole part I.

1. Let σ, τ be two stopping times with respect to $\mathbb{F}$. Then it is always true that
   (a) $\tau \vee \sigma$ is not a stopping time.
   (b) $\sigma + \tau$ is a stopping time.
   (c) $(\tau - 1)^+$ is a stopping time.
2. Let Q^* be an equivalent martingale measure for S^1. Then
   (a) the unique price process of an attainable payoff is a Q^*-martingale.
   (b) there is an admissible trading strategy whose gains process at time T is strictly positive with probability 1.
   (c) the market is complete.
3. Let Q^* be the unique equivalent martingale measure for S^1. Then it is always true that
   (a) $Q^* \neq P$.
   (b) $\lambda Q^* + (1-\lambda)P \approx P$ for every $\lambda \in [0,1]$.
   (c) $\left(E\left[\frac{\mathrm{d}P}{\mathrm{d}Q^*}\middle|\mathcal{F}_k\right]\, S_k^1\right)_{k=0}^{T}$ is a $(P,\mathbb{F})$-martingale.
4. Consider a binomial model with $T = 5$, $r = 1/2$, $\widetilde{S}_0^1 = 1$, and $\widetilde{S}_k^1 = Y^k$, where $P[Y=2] = 1/3$ and $P[Y = 1/2] = 2/3$. Consider the strategy $\varphi := (\varphi^0, \vartheta)$, where $\varphi^0 = (\varphi_k^0)_{k=0}^T$, $\vartheta = (\vartheta_k)_{k=0}^T$ with $\varphi_0^0 = \vartheta_0 = 0$ and

   $$\varphi_k^0 := -\mathbb{1}_{\{k\le 3\}} \qquad \text{and} \qquad \vartheta_k := \mathbb{1}_{\{k\le 3\}},$$

   for all $k = 1,\dots,T$. Which of the following assertions is true?
   (a) The market is arbitrage free and the strategy is not self-financing.
   (b) The market is not arbitrage free and the strategy is self-financing.
   (c) None of the previous answers is correct.
5. Let σ, τ be two stopping times with respect to $\mathbb{F}$. Then it is always true that
   (a) the product $\sigma\tau$ is a stopping time.
   (b) the sum $\sigma + \tau$ is a stopping time.
   (c) the difference $\sigma - \tau$ is a stopping time.

6. Assume the market is arbitrage-free and complete. Then

   (a) for every contingent claim H, the cost of replicating H is unique.

   (b) every self-financing portfolio is admissible.

   (c) every admissible portfolio is self-financing.

7. Suppose that P is a probability measure and S^1 is a $(P, \mathcal{F})$-submartingale. Then S^1 is a $(P, \mathcal{F})$-martingale **if**

   (a) $S_0^1 \leq E_P[S_T^1]$.

   (b) $S_\sigma^1 \geq E_P[S_\tau^1 \mid \mathcal{F}_\sigma]$ for every pair of stopping times σ, τ satisfying $\sigma \leq \tau$.

   (c) the stochastic integral process $\varphi \cdot S^1$ is a $(P, \mathcal{F})$-supermartingale for some predictable process φ.

8. Assume that the market is free of arbitrage. Then it is always true that

   (a) the set of equivalent martingale measures contains at most one element.

   (b) the set of equivalent martingale measures contains exactly one element.

   (c) the set of equivalent martingale measures contains at least one element.

9. Let (S^0, S^1) be an arbitrage-free incomplete market. Which of the following statements is **not** true?

   (a) The set of all equivalent martingale measures for S^1 is uncountably infinite.

   (b) There are multiple ways to assign a price process $V^H = (V_k^H)_{k=0,1,\ldots,T}$ to an unattainable payoff $H \in L_+^0(\mathcal{F}_T)$, such that the market (S^0, S^1, V^H) is arbitrage-free.

   (c) We cannot replicate any payoff $H \in L_+^0(\mathcal{F}_T)$ by trading in S^0 and S^1.

10. Let Q be an equivalent martingale measure for (S^0, S^1). Then we always have that

    (a) both $(\widetilde{S}^0, \widetilde{S}^1)$ and (S^0, S^1) are free of arbitrage.

    (b) $E_Q\left[\widetilde{S}_k^1 \middle| \mathcal{F}_{k-1}\right] = \widetilde{S}_{k-1}^1$ for all $k \in \{1, 2, \ldots, T\}$.

    (c) $(\widetilde{S}^0, \widetilde{S}^1)$ is a complete market.

11. Let $M = (M_k)_{k=0,1,\ldots,T}$ be a local $(P, \mathbb{F})$-martingale. Then

    (a) since we are in finite discrete time, M is even a true $(P, \mathbb{F})$-martingale.

    (b) there exists an $\mathbb{F}$-stopping time τ such that M^τ is a local $(P, \mathbb{F})$-martingale.

    (c) M is not integrable.

12. Let $Z = (Z_k)_{k=0,1,\ldots,T}$ be the density process of $Q \approx P$ with respect to P. Then

    (a) Z is s strictly positive $(P, \mathbb{F})$-martingale.

    (b) $\frac{1}{Z}$ is a $(P, \mathbb{F})$-martingale.

    (c) ZS^1 is a $(Q, \mathbb{F})$-martingale if and only if S^1 is $(P, \mathbb{F})$-martingale.

13. Let $\varphi = (\varphi^0, \vartheta)$ be an arbitrage strategy for $(\widetilde{S}^0, \widetilde{S}^1)$. Then

    (a) $G_k(\vartheta) \geq 0$ P-a.s. for all $k \in \{0, 1, \ldots, T\}$.

    (b) $V(\varphi) = \varphi_0^0 + G(\vartheta)$ P-a.s.

    (c) $V_0(\varphi) > 0$ P-a.s.

14. Which of the following conditions does **not** imply that $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free?
    (a) The set of ELMMs for (S^0, S^1) is non-empty.
    (b) S^1 is a $(Q, \mathbb{F})$-martingale for some probability measure $Q \approx P$.
    (c) There exists an EMM for $(\widetilde{S}^0, \widetilde{S}^1)$.

15. Which of the following statements is true?
    (a) Let $D > 0$ be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$. Then $\frac{D}{E[D]}$ is the density (Radon–Nikodým derivative) of some probability measure $Q \approx P$.
    (b) Every bounded $(P, \mathbb{F})$-martingale is P-a.s. constant.
    (c) There exists no $(P, \mathbb{F})$-submartingale which is also a $(P, \mathbb{F})$-martingale.

16. Let $(\widetilde{S}^0, \widetilde{S}^1)$ follow the binomial model. Then
    (a) $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free and complete.
    (b) $(\widetilde{S}^0, \widetilde{S}^1)$ is complete, provided that it is arbitrage-free.
    (c) $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free.

17. Let M be a local $(P, \mathbb{F})$-martingale and A be a process of finite variation. Then
    (a) $[M, A] = 0$ P-a.s.
    (b) $[M, A] = 0$ P-a.s. if and only if both M and A are continuous.
    (c) $[M, A] = 0$ P-a.s. if A is continuous.

18. Let $Z = (Z_t)_{t \in [0,T]}$ for some $T \in (0, \infty)$ be the density process of $Q \approx P$ on $\mathcal{F}_T$ with respect to P. Which of the following is true?
    (a) $Z = Z_0 \mathcal{E}(L)$ for some continuous local $(P, \mathbb{F})$-martingale $L = (L_t)_{t \in [0,T]}$.
    (b) $Z = Z_0 \mathcal{E}(L)$ for some local $(P, \mathbb{F})$-martingale $L = (L_t)_{t \in [0,T]}$.
    (c) $Z = Z_0 \mathcal{E}(L)$ for some local $(Q, \mathbb{F})$-martingale $L = (L_t)_{t \in [0,T]}$.

19. Which of the following statements about W is **not** true?
    (a) P-a.a. paths of W have infinite 1-variation.
    (b) W is a $(P, \mathbb{F})$-semimartingale.
    (c) W is the unique continuous process with zero mean and normally distributed increments.

20. Let X be a $(P, \mathbb{F})$-semimartingale and let $Q \overset{loc}{\approx} P$. Then
    (a) $f(X)$ is a $(Q, \mathbb{F})$-semimartingale for any C^∞-function f.
    (b) X is a $(Q, \mathbb{F})$-martingale.
    (c) $f(X)$ is a $(P, \mathbb{F})$-semimartingale for any measurable function f.

21. The set of attainable payoffs:
    (a) is closed under addition.
    (b) is closed under scalar multiplication.
    (c) consists of integrable random variables.

22. Consider a financial market in finite discrete time. Which assumption about the market justifies the formula $\Delta C_{k+1} = \Delta \varphi^0_{k+1} + \Delta \vartheta_{k+1} S^1_{k+1}$?

(a) The investor is small.
(b) Trading strategies are unrestricted.
(c) None of the above.

23. Suppose S^1 is a submartingale. Then:
    (a) The market is arbitrage-free.
    (b) For ϑ admissible, $\vartheta \cdot S^1$ is a submartingale.
    (c) $(S^1 - 1)^+ - \frac{1}{2}(S^1 - 1)^-$ is a submartingale.

24. Let X, Y be semimartingales. Then:
    (a) XY is a martingale, if X and Y are martingales.
    (b) XY is a martingale, if X is a martingale and Y has finite variation.
    (c) XY has finite variation, if X and Y have finite variation.

25. Which of the following is a stopping time?
    (a) $\tau = \sup\{t \geq 0 : W_t \geq t\}$
    (b) $\tau = \inf\{t \geq 1 : \int_1^t \operatorname{sgn}(W_{s-1}) dW_s > 2\}$, where $\operatorname{sgn}(z) = \mathbb{1}_{z>0} - \mathbb{1}_{z<0}$.
    (c) $\tau = \inf\{t \geq 0 : W_{t^2} \geq 1\}$

26. Let $Q \approx P$. Then:
    (a) If X is a P-submartingale, it is a Q-submartingale.
    (b) If X is a continuous P-martingale, then $X^2 - [X]$ is a continuous local Q-martingale.
    (c) $[W]_t = t$ holds Q-almost surely.

27. In the Black-Scholes model, let $f(\widetilde{S}_T)$ be a payoff, for a smooth function f. What is the PDE associated with its value process $\widetilde{v}(t, \widetilde{S}_t)$?
    (a)
    $$0 = \frac{\partial \widetilde{v}}{\partial t} + r\widetilde{x}\frac{\partial \widetilde{v}}{\partial \widetilde{x}} + \frac{1}{2}\sigma^2 \widetilde{x}^2 \frac{\partial^2 \widetilde{v}}{\partial \widetilde{x}^2} - r\widetilde{v}, \quad \widetilde{v}(T, \widetilde{x}) = f(\widetilde{x})$$
    (b)
    $$0 = -\frac{\partial \widetilde{v}}{\partial t} + r\widetilde{x}\frac{\partial \widetilde{v}}{\partial \widetilde{x}} + \frac{1}{2}\sigma^2 \widetilde{x}^2 \frac{\partial^2 \widetilde{v}}{\partial \widetilde{x}^2} - r\widetilde{v}, \quad \widetilde{v}(T, \widetilde{x}) = f(\widetilde{x})$$
    (c)
    $$f = -\frac{\partial \widetilde{v}}{\partial t} + r\widetilde{x}\frac{\partial \widetilde{v}}{\partial \widetilde{x}} + \frac{1}{2}\sigma^2 \widetilde{x}^2 \frac{\partial^2 \widetilde{v}}{\partial \widetilde{x}^2} - r\widetilde{v}, \quad \widetilde{v}(T, \widetilde{x}) = 0$$

28. Which of the following statements is **not** true about the binomial model?
    (1) The market is complete if it is arbitrage-free.
    (2) Every strategy is admissible.
    (3) Every strategy is self-financing.

29. Let M be an adapted process. Which of the following does **not** imply that M is a martingale?
    (1) M is a supermartingale such that $E[M_k]$ is an increasing sequence.
    (2) For each $k = 1, \ldots, T$, $E[M_k - M_{k-1} \mid \mathcal{F}_{k-1}] = 0$.
    (3) M is a bounded local martingale.

30. Let M be a local $(P, \mathbb{F})$-martingale and A be a process of finite variation. Then
    (1) $H \bullet M$ is a martingale, if H is bounded and predictable.
    (2) $H \bullet A$ has finite variation, if H is locally bounded and predictable.
    (3) $H \bullet M$ is a martingale, if M is a martingale and H is locally bounded and predictable.
31. Which of the following equations has a unique solution $Z = (Z_t)_{t\in[0,T]}$?
    (1) $dZ_t = Z_t(3dt + dW_t)$
    (2) $[Z]_t = t$, for Z a continuous local martingale with $Z_0 = 0$.
    (3) $Z_t = \int_0^t (2 + \cos(Z_s))dW_s$
32. Suppose that $Q \approx P$ with density $\frac{dQ}{dP} = \exp(W_t - t/2)$. Which of the following is **not** true?
    (1) $W_t^2 - t$ is a Q-martingale.
    (2) $\exp(W_t - 3t/2)$ is a Q-martingale.
    (3) $f(W_t)$ is a $(Q, \mathbb{F})$-semimartingale, for $f \in C^2(\mathbb{R})$.
33. Consider the Black-Scholes model. Then:
    (1) The price of a European put option increases, if the strike is decreased.
    (2) The price of a European put option increases, if the interest rate is increased.
    (3) None of the above.

Solution 14.3

1. (2)
2. (1)
3. (2)
4. (3)
5. (2)
6. (1)
7. (2)
8. (3)
9. (3)
10. (1)
11. (2)
12. (1)
13. (2)
14. (3)
15. (1)
16. (2)

17. (3)
18. (2)
19. (3)
20. (1)
21. (1)
22. (1)
23. (3)
24. (3)
25. (2)
26. (3)
27. (1)
28. (3)
29. (2)
30. (2)
31. (3)
32. (1)
33. (3)