

Mathematical Foundations for Finance

Exercise sheet 2

Please upload your solutions until Wednesday, 06/10/2021, 12:00 using the link on the course website.

Exercise 2.1 Let us assume the basic multiplicative model for our financial market $(\tilde{S}^0, \tilde{S}^1)$. We start on a probability space (Ω, \mathcal{F}, P) with random variables $r_1, \dots, r_T > -1$ and $Y_1, \dots, Y_T > 0$ for a $T \in \mathbb{N}$. Define for $k = 0, \dots, T$

$$\tilde{S}_k^0 := \prod_{j=1}^k (1 + r_j), \quad \tilde{S}_k^1 := S_0^1 \prod_{j=1}^k Y_j,$$

with a constant $S_0^1 > 0$.

- (a) A natural filtration to use in this model is the filtration generated by $Y = (Y_k)_{k=1, \dots, T}$ and $r = (r_k)_{k=1, \dots, T}$, i.e. the one given by

$$\begin{aligned} \mathcal{F}'_0 &= \{\emptyset, \Omega\}, \\ \mathcal{F}'_k &= \sigma(Y_1, \dots, Y_k, r_1, \dots, r_k) \quad \text{for } k = 1, \dots, T. \end{aligned}$$

Show that if one assumes r to be predictable with respect to this filtration, then we have that $\mathcal{F}'_k = \mathcal{F}_k := \sigma(\tilde{S}_0^1, \tilde{S}_1^1, \dots, \tilde{S}_k^1)$ for all $k = 0, \dots, T$.

(Hint: if \mathcal{A} and \mathcal{B} are two collections of subsets of Ω , then $\sigma(\mathcal{A} \cup \mathcal{B}) = \sigma(\sigma(\mathcal{A}) \cup \sigma(\mathcal{B}))$)

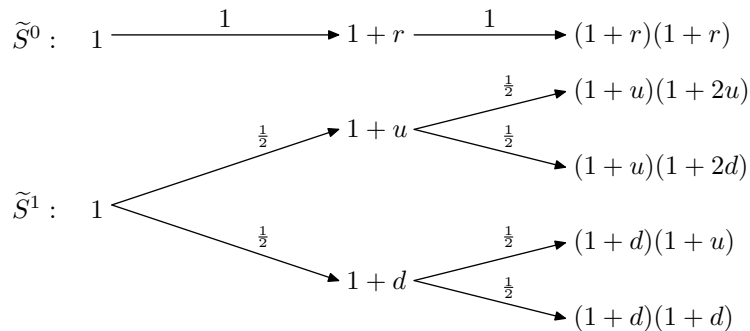
- (b) Recall that we call a strategy $\varphi = (\varphi^0, \vartheta)$ self-financing if its discounted cost process $C(\varphi)$ is constant over time. Show that the notion of self-financing strategy does not depend on whether we work with the discounted price processes S^0 and S^1 or the undiscounted processes \tilde{S}^0 and \tilde{S}^1 , i.e. show that the discounted cost process $C(\varphi)$ is constant over time if and only if the undiscounted cost process $\tilde{C}(\varphi)$, determined by

$$\Delta \tilde{C}_{k+1}(\varphi) := \tilde{C}_{k+1}(\varphi) - \tilde{C}_k(\varphi) = (\varphi_{k+1}^0 - \varphi_k^0) \tilde{S}_k^0 + (\vartheta_{k+1} - \vartheta_k) \tilde{S}_k^1,$$

is constant over time.

- (c) Use the result in (b) to conclude that the notion of self-financing strategy is numeraire-invariant, i.e. that it does not matter for this definition whether the discounted price processes are defined as $S^0 := \tilde{S}^0 / \tilde{S}^0$ and $S^1 := \tilde{S}^1 / \tilde{S}^0$, or $\bar{S}^0 := \tilde{S}^0 / \tilde{S}^1$ and $\bar{S}^1 := \tilde{S}^1 / \tilde{S}^1$.

Exercise 2.2 Consider a financial market $(\tilde{S}^0, \tilde{S}^1)$ given by the following trees, where the numbers beside the branches denote transition probabilities.



Intuitively, this means that the volatility of \tilde{S}^1 increases if the stock price increases in the first period. Assume that $u, r \geq 0$ and $-0.5 < d \leq 0$.

- Construct for this setup a multiplicative model consisting of a probability space (Ω, \mathcal{F}, P) , a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, two random variables Y_1 and Y_2 and two adapted stochastic processes \tilde{S}^0 and \tilde{S}^1 such that $\tilde{S}_k^1 = \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$.
- For which values of u and d are Y_1 and Y_2 *uncorrelated*?
- For which values of u and d are Y_1 and Y_2 *independent*?
- For which values of u, r and d is the discounted stock process S^1 a P -martingale?

Exercise 2.3 Consider for a finite time horizon $T \geq 2$ a financial market $(\tilde{S}^0, \tilde{S}^1)$ consisting of a bank account and one stock defined on a probability space (Ω, \mathcal{F}, P) . Assume that $\tilde{S}_0^1 = 1$ and $\tilde{S}_k^1 > 0$ P -a.s. for all $k = 0, \dots, T$. Fix thresholds $0 < \ell < 1 < u$ and define

$$\begin{aligned}\sigma(\omega) &:= \inf\{k = 0, \dots, T : S_k^1(\omega) \leq \ell\} \wedge T, \\ \tau(\omega) &:= \inf\{k = \sigma(\omega), \dots, T : S_k^1(\omega) \geq u\} \wedge T,\end{aligned}$$

where $\inf \emptyset = +\infty$ as usual. Moreover, for $k = 0, \dots, T$ define

$$\vartheta_k(\omega) := \mathbb{1}_{\{\sigma(\omega) < k \leq \tau(\omega)\}}.$$

Finally define the filtration $\mathbb{F} = (\mathcal{F}_k)_{0 \leq k \leq T}$ by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_k = \sigma(\tilde{S}_i^1, i \leq k)$.

- Show that σ and τ are *stopping times*, i.e. that for all $k = 0, \dots, T$, we have

$$\{\sigma \leq k\}, \{\tau \leq k\} \in \mathcal{F}_k.$$

- Show that ϑ is a predictable process with $\vartheta_0 = \vartheta_1 = 0$.
- Construct φ^0 such that $\varphi = (\varphi^0, \vartheta)$ is a self-financing strategy with $V_0(\varphi) = 0$ and derive a formula for the (discounted) value process $V(\varphi)$ involving only the discounted stock price S^1 and the stopping times σ and τ .
- Describe the trading strategy φ in words.