

Mathematical Foundations for Finance

Exercise sheet 2

Please upload your solutions until Wednesday, 06/10/2021, 12:00 using the link on the course website.

Exercise 2.1 Let us assume the basic multiplicative model for our financial market $(\tilde{S}^0, \tilde{S}^1)$. We start on a probability space (Ω, \mathcal{F}, P) with random variables $r_1, \dots, r_T > -1$ and $Y_1, \dots, Y_T > 0$ for a $T \in \mathbb{N}$. Define for $k = 0, \dots, T$

$$\tilde{S}_k^0 := \prod_{j=1}^k (1 + r_j), \quad \tilde{S}_k^1 := S_0^1 \prod_{j=1}^k Y_j,$$

with a constant $S_0^1 > 0$.

- (a) A natural filtration to use in this model is the filtration generated by $Y = (Y_k)_{k=1, \dots, T}$ and $r = (r_k)_{k=1, \dots, T}$, i.e. the one given by

$$\begin{aligned} \mathcal{F}'_0 &= \{\emptyset, \Omega\}, \\ \mathcal{F}'_k &= \sigma(Y_1, \dots, Y_k, r_1, \dots, r_k) \quad \text{for } k = 1, \dots, T. \end{aligned}$$

Show that if one assumes r to be predictable with respect to this filtration, then we have that $\mathcal{F}'_k = \mathcal{F}_k := \sigma(\tilde{S}_0^1, \tilde{S}_1^1, \dots, \tilde{S}_k^1)$ for all $k = 0, \dots, T$.

(Hint: if \mathcal{A} and \mathcal{B} are two collections of subsets of Ω , then $\sigma(\mathcal{A} \cup \mathcal{B}) = \sigma(\sigma(\mathcal{A}) \cup \sigma(\mathcal{B}))$)

- (b) Recall that we call a strategy $\varphi = (\varphi^0, \vartheta)$ self-financing if its discounted cost process $C(\varphi)$ is constant over time. Show that the notion of self-financing strategy does not depend on whether we work with the discounted price processes S^0 and S^1 or the undiscounted processes \tilde{S}^0 and \tilde{S}^1 , i.e. show that the discounted cost process $C(\varphi)$ is constant over time if and only if the undiscounted cost process $\tilde{C}(\varphi)$, determined by

$$\Delta \tilde{C}_{k+1}(\varphi) := \tilde{C}_{k+1}(\varphi) - \tilde{C}_k(\varphi) = (\varphi_{k+1}^0 - \varphi_k^0) \tilde{S}_k^0 + (\vartheta_{k+1} - \vartheta_k) \tilde{S}_k^1,$$

is constant over time.

- (c) Use the result in (b) to conclude that the notion of self-financing strategy is numeraire-invariant, i.e. that it does not matter for this definition whether the discounted price processes are defined as $S^0 := \tilde{S}^0 / \tilde{S}^0$ and $S^1 := \tilde{S}^1 / \tilde{S}^0$, or $\bar{S}^0 := \tilde{S}^0 / \tilde{S}^1$ and $\bar{S}^1 := \tilde{S}^1 / \tilde{S}^1$.

Solution 2.1

- (a) As suggested in the Remark on page 7 of the lecture notes, we proceed by induction:

1. We show that $\mathcal{F}'_k = \mathcal{F}_k$ for $k = 0$:

We clearly have that $\mathcal{F}'_0 = \mathcal{F}_0$ since \tilde{S}_0^1 is a constant random variable that generates the trivial σ -algebra $\{\emptyset, \Omega\}$.

2. The induction step - $\mathcal{F}'_k = \mathcal{F}_k \implies \mathcal{F}'_{k+1} = \mathcal{F}_{k+1}$:

We have that

$$\begin{aligned}\mathcal{F}'_{k+1} &= \sigma(\sigma(r_1, \dots, r_k, Y_1, \dots, Y_k) \cup \sigma(r_{k+1}) \cup \sigma(Y_{k+1})) \\ &= \sigma(\sigma(\tilde{S}_1^1, \dots, \tilde{S}_k^1) \cup \sigma(r_{k+1}) \cup \sigma(Y_{k+1})) \\ &= \sigma(\sigma(\tilde{S}_1^1, \dots, \tilde{S}_{k+1}^1) \cup \sigma(r_{k+1}))\end{aligned}$$

where for the second equality we have used our induction hypothesis that $\mathcal{F}'_k = \mathcal{F}_k$ and for the third we used the hint and the fact that

$$\sigma(\tilde{S}_1^1, \dots, \tilde{S}_k^1, Y_{k+1}) = \sigma(\tilde{S}_1^1, \dots, \tilde{S}_k^1, \tilde{S}_{k+1}^1)$$

since $\tilde{S}_{k+1}^1 = \tilde{S}_k^1 Y_{k+1}$. Furthermore, since r_{k+1} is \mathcal{F}'_k -measurable, and thus also \mathcal{F}_k -measurable by our induction hypothesis, we must have that $\sigma(r_{k+1}) \subseteq \mathcal{F}_k$. This gives

$$\mathcal{F}'_{k+1} = \sigma(\mathcal{F}_{k+1} \cup \sigma(r_{k+1})) = \sigma(\mathcal{F}_{k+1}) = \mathcal{F}_{k+1}$$

and concludes the proof.

- (b) Using the definition of the incremental cost for a strategy φ from the lecture, $\Delta C_{k+1}(\varphi) = C_{k+1}(\varphi) - C_k(\varphi)$, we get that

$$C_{k+1}(\varphi) = \Delta C_{k+1}(\varphi) + C_k(\varphi) = C_0(\varphi) + \sum_{j=1}^{k+1} \Delta C_j(\varphi).$$

Therefrom it is clear that the cost process $C_k(\varphi)$ is constant over time (and equal to the initial investment into the bank account, φ_0^0) if and only if we have for each k that $\Delta C_{k+1}(\varphi) = 0$ P -a.s. Since we have that

$$\Delta C_k(\varphi) = (\varphi_{k+1}^0 - \varphi_k^0)S_k^0 + (\vartheta_{k+1} - \vartheta_k)S_k^1,$$

the self-financing condition boils down to

$$0 = (\varphi_{k+1}^0 - \varphi_k^0)S_k^0 + (\vartheta_{k+1} - \vartheta_k)S_k^1$$

P -a.s. for each k . Multiplying both sides of the equation by \tilde{S}_k^0 we obtain the same condition for the undiscounted prices, i.e.

$$0 = (\varphi_{k+1}^0 - \varphi_k^0)\tilde{S}_k^0 + (\vartheta_{k+1} - \vartheta_k)\tilde{S}_k^1 = \Delta \tilde{C}_{k+1}(\varphi).$$

Since we also have that

$$\tilde{C}_{k+1}(\varphi) = \tilde{C}_0(\varphi) + \sum_{j=1}^{k+1} \Delta \tilde{C}_j(\varphi),$$

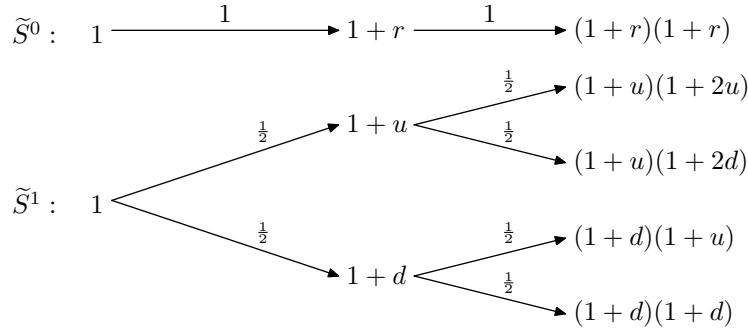
this indeed shows that the discounted cost process is P -a.s. constant if and only if the undiscounted process is.

- (c) We have shown in (b) that a strategy φ is self-financing if and only if we have that

$$(\varphi_{k+1}^0 - \varphi_k^0)\tilde{S}_k^0 + (\vartheta_{k+1} - \vartheta_k)\tilde{S}_k^1 = 0.$$

P -a.s. for all k . In order to express this condition in either version of the discounted processes S^0 and S^1 , it is enough to divide both sides of the equation by \tilde{S}_k^1 or \tilde{S}_k^0 . Since $\tilde{S}_k^1, \tilde{S}_k^0 > 0$, the self-financing condition still holds after this operation.

Exercise 2.2 Consider a financial market $(\tilde{S}^0, \tilde{S}^1)$ given by the following trees, where the numbers beside the branches denote transition probabilities.



Intuitively, this means that the volatility of \tilde{S}^1 increases if the stock price increases in the first period. Assume that $u, r \geq 0$ and $-0.5 < d \leq 0$.

- (a) Construct for this setup a multiplicative model consisting of a probability space (Ω, \mathcal{F}, P) , a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, two random variables Y_1 and Y_2 and two adapted stochastic processes \tilde{S}^0 and \tilde{S}^1 such that $\tilde{S}_k^1 = \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$.
- (b) For which values of u and d are Y_1 and Y_2 *uncorrelated*?
- (c) For which values of u and d are Y_1 and Y_2 *independent*?
- (d) For which values of u, r and d is the discounted stock process S^1 a P -martingale?

Solution 2.2

- (a) We construct the canonical model for this setup, a path space. Let $\Omega := \{-1, 1\}^2$, take $\mathcal{F} := 2^\Omega$ and define P by

$$P[\{(x_1, x_2)\}] := p_{x_1} p_{x_1, x_2},$$

where $p_1 = p_{-1} := 1/2$ and $p_{1,1} = p_{1,-1} = p_{-1,1} = p_{-1,-1} := 1/2$. Next, define Y_1 and Y_2 by $Y_1((1, 1)) = Y_1((1, -1)) := 1 + u$, $Y_1((-1, 1)) = Y_1((-1, -1)) := 1 + d$ and $Y_2((1, 1)) := 1 + 2u$, $Y_2((1, -1)) := 1 + 2d$, $Y_2((-1, 1)) := 1 + u$, $Y_2((-1, -1)) := 1 + d$. Finally, define \tilde{S}^0 and \tilde{S}^1 by $\tilde{S}_k^0 := (1 + r)^k$ and $\tilde{S}_k^1 := \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$ and set $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \sigma(Y_1) = \{\emptyset, \{(1, 1), (1, -1)\}, \{(-1, 1), (-1, -1)\}, \Omega\}$ and $\mathcal{F}_2 := \sigma(Y_1, Y_2) = 2^\Omega = \mathcal{F}$.

- (b) Y_1 and Y_2 are uncorrelated if and only if $E[Y_1 Y_2] = E[Y_1] E[Y_2]$. Set $c := (u + d)/2$ to simplify the notation. Then we have

$$E[Y_1] = 1 + c \quad \text{and} \quad E[Y_2] = 1 + \frac{3}{2}c,$$

$$E[Y_1 Y_2] = \frac{1 + u}{2}(1 + 2c) + \frac{1 + d}{2}(1 + c) = (1 + c)^2 + \frac{1 + u}{2}c.$$

Hence, we have

$$E[Y_1 Y_2] - E[Y_1] E[Y_2] = (1 + c)^2 + \frac{1 + u}{2}c - \left((1 + c)^2 + (1 + c)\frac{c}{2} \right)$$

$$= (u - c)\frac{c}{2}.$$

Since $d \leq 0 \leq u$, we have

$$(u - c)\frac{c}{2} = 0 \quad \Leftrightarrow \quad c = 0 \quad \text{or} \quad u - c = 0 \quad \Leftrightarrow \quad d = -u.$$

In conclusion, Y_1 and Y_2 are uncorrelated if and only if $d = -u$.

- (c) Since independence of two random variables implies that they are uncorrelated, we only have to consider the case in which $u = -d$. If $u = d = 0$, Y_1 and Y_2 are both constant random variables and hence independent. Otherwise, if $u > 0$ we have on the one hand

$$P[Y_1 = 1 + u, Y_2 = 1 + u] = 0$$

and on the other hand

$$P[Y_1 = 1 + u]P[Y_2 = 1 + u] = 1/2 \cdot 1/4 = 1/8 \neq 0,$$

showing that in this case Y_1 and Y_2 are not independent. In conclusion, Y_1 and Y_2 are independent if and only if $u = d = 0$.

Note: If $d = -u$ and $u \neq 0$, then Y_1 and Y_2 are uncorrelated but **not** independent.

- (d) S^1 is a P -martingale if and only if

$$E[S_1^1 | \mathcal{F}_0] = S_0^1 \quad P\text{-a.s.} \quad \text{and} \quad E[S_2^1 | \mathcal{F}_1] = S_1^1 \quad P\text{-a.s.} \quad (1)$$

If $u = d = 0$, it is straightforward to check that S^1 is a P -martingale if and only if $r = 0$. Next, assume that $u > d$. Since \mathcal{F}_0 is trivial, $\mathcal{F}_1 = \sigma(Y_1)$ and $Y_1 > 0$, (1) is equivalent to

$$E[Y_1] = 1 + r \quad \text{and} \quad E[Y_2 | Y_1] = 1 + r \quad P\text{-a.s.}$$

Since Y_1 only takes two values, this is equivalent to

$$E[Y_1] = 1 + r \quad \text{and} \quad E[Y_2 | Y_1 = 1 + u] = 1 + r \quad \text{and} \quad E[Y_2 | Y_1 = 1 + d] = 1 + r.$$

This is equivalent to the linear system

$$\begin{aligned} 1 + (u + d)/2 &= 1 + r, \\ 1 + u + d &= 1 + r, \\ 1 + (u + d)/2 &= 1 + r. \end{aligned}$$

Subtracting the first from the second equation yields $(u + d)/2 = 0$, which in turn implies $r = 0$. In conclusion, S^1 is a P -martingale if and only if $r = 0$ and $d = -u$.

Exercise 2.3 Consider for a finite time horizon $T \geq 2$ a financial market $(\tilde{S}^0, \tilde{S}^1)$ consisting of a bank account and one stock defined on a probability space (Ω, \mathcal{F}, P) . Assume that $\tilde{S}_0^1 = 1$ and $\tilde{S}_k^1 > 0$ P -a.s. for all $k = 0, \dots, T$. Fix thresholds $0 < \ell < 1 < u$ and define

$$\begin{aligned} \sigma(\omega) &:= \inf\{k = 0, \dots, T : S_k^1(\omega) \leq \ell\} \wedge T, \\ \tau(\omega) &:= \inf\{k = \sigma(\omega), \dots, T : S_k^1(\omega) \geq u\} \wedge T, \end{aligned}$$

where $\inf \emptyset = +\infty$ as usual. Moreover, for $k = 0, \dots, T$ define

$$\vartheta_k(\omega) := \mathbb{1}_{\{\sigma(\omega) < k \leq \tau(\omega)\}}.$$

Finally define the filtration $\mathbb{F} = (\mathcal{F}_k)_{0 \leq k \leq T}$ by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_k = \sigma(\tilde{S}_i^1, i \leq k)$.

- (a) Show that σ and τ are *stopping times*, i.e. that for all $k = 0, \dots, T$, we have

$$\{\sigma \leq k\}, \{\tau \leq k\} \in \mathcal{F}_k.$$

- (b) Show that ϑ is a predictable process with $\vartheta_0 = \vartheta_1 = 0$.

- (c) Construct φ^0 such that $\varphi = (\varphi^0, \vartheta)$ is a self-financing strategy with $V_0(\varphi) = 0$ and derive a formula for the (discounted) value process $V(\varphi)$ involving only the discounted stock price S^1 and the stopping times σ and τ .
- (d) Describe the trading strategy φ in words.

Solution 2.3

- (a) We have $\{\sigma \leq 0\} = \emptyset \in \mathcal{F}_0$ and $\{\sigma \leq T\} = \Omega \in \mathcal{F}_T$. For $k = 1, \dots, T - 1$ we have

$$\{\sigma \leq k\} = \bigcup_{j=1}^k \{S_j^1 \leq \ell\} \in \mathcal{F}_k,$$

since for $1 \leq j \leq k$ we have $\{S_j^1 \leq \ell\} \in \mathcal{F}_j \subset \mathcal{F}_k$ (because S^1 is adapted) and a finite union of sets in \mathcal{F}_k is in \mathcal{F}_k by the definition of σ -algebra.

We have $\{\tau \leq 0\} = \emptyset \in \mathcal{F}_0$, $\{\tau \leq 1\} = \emptyset \in \mathcal{F}_1$ and $\{\tau \leq T\} = \Omega \in \mathcal{F}_T$. For $k = 2, \dots, T - 1$ we have

$$\{\tau \leq k\} = \bigcup_{1 \leq i < j \leq k} \{S_i^1 \leq \ell, S_j^1 \geq u\} \in \mathcal{F}_k,$$

since for $1 \leq i < j \leq k$ we have

$$\{S_i^1 \leq \ell, S_j^1 \geq u\} = \{S_i^1 \leq \ell\} \cap \{S_j^1 \geq u\}, \{S_i^1 \leq \ell\} \in \mathcal{F}_i \subset \mathcal{F}_k$$

and $\{S_j^1 \geq u\} \in \mathcal{F}_j \subset \mathcal{F}_k$ (because S^1 is adapted) and a finite union of sets in \mathcal{F}_k is in \mathcal{F}_k by the definition of σ -algebra.

- (b) Since $\sigma \geq 1$ P -a.s., it follows immediately from the definition of ϑ that $\vartheta_0 = \vartheta_1 = 0$. It remains to show that for all $k = 2, \dots, T$ we have $\{\vartheta_k = 1\} \in \mathcal{F}_{k-1}$. Indeed, for $k = 2, \dots, T$ we have

$$\{\vartheta_k = 1\} = \{\sigma \leq k - 1, \tau \geq k\} = \{\sigma \leq k - 1\} \setminus \{\tau \leq k - 1\} \in \mathcal{F}_{k-1},$$

since σ and τ are stopping times.

- (c) First, we calculate the (discounted) gains process associated with ϑ , which is for $k = 0, \dots, T$ given by

$$G_k(\vartheta) = \sum_{j=1}^k \mathbf{1}_{\{\sigma < j \leq \tau\}} \Delta S_j^1 = S_{\tau \wedge k}^1 - S_{\sigma \wedge k}^1 \quad P\text{-a.s.}$$

By formula (2.8) in the lecture notes, the trading strategy $\varphi = (\varphi^0, \vartheta)$ is self-financing if and only if for $k = 0, \dots, T$ we have

$$V_k(\varphi) = V_0(\varphi) + G_k(\vartheta) \quad P\text{-a.s.}$$

Using that $V_k(\varphi) = \varphi_k^0 + \vartheta_k S_k^1$ by definition and $V_0(\varphi) = 0$ by assumption, we get $\varphi_0^0 = 0$ and for $k = 1, \dots, T$ we have P -a.s.

$$\begin{aligned} \varphi_k^0 &= G_k(\vartheta) - \vartheta_k S_k^1 = S_{\tau \wedge k}^1 - S_{\sigma \wedge k}^1 - \mathbf{1}_{\{\sigma < k \leq \tau\}} S_k^1 \\ &= -S_{\sigma}^1 \mathbf{1}_{\{\sigma < k \leq \tau\}} + (S_{\tau}^1 - S_{\sigma}^1) \mathbf{1}_{\{k > \tau\}}. \end{aligned}$$

Moreover, we clearly have for $k = 0, \dots, T$

$$V_k(\varphi) = G_k(\vartheta) = S_{\tau \wedge k}^1 - S_{\sigma \wedge k}^1 \quad P\text{-a.s.}$$

- (d) This strategy can be described as a “buy low and sell high” strategy. When the discounted price of the stock falls below ℓ , one borrows money to buy a share of the stock. As soon as the discounted price of the stock is above u , one sells the share, pays back the loan and stores the difference between buying and selling price in the bank account.