

Mathematical Foundations for Finance

Exercise sheet 3

Please upload your solutions until Wednesday, 13/10/2021, 12:00 using the link on the course website.

Exercise 3.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ and $X = (X_k)_{k=0,1,\dots,T}$ a martingale with respect to \mathbb{F} and P .

- (a) Show that for a bounded, convex function $f: \mathbb{R} \rightarrow \mathbb{R}$ the process $f(X) = (f(X_k))_{k=0,1,\dots,T}$ is a submartingale with respect to \mathbb{F} and P . What can you say if f is not bounded?

Hint: Any convex function on \mathbb{R}^n is continuous on the interior of the set where it is finite.

- (b) Let $\vartheta = (\vartheta_k)_{k=0,1,\dots,T}$ with $\vartheta_0 = 0$ be a bounded, nonnegative, \mathbb{F} -predictable process. Show that the stochastic integral process $\vartheta \bullet f(X)$ defined by

$$\vartheta \bullet f(X)_k = \sum_{j=1}^k \vartheta_j \Delta f(X_j)$$

is a submartingale with respect to \mathbb{F} and P .

Hint: We have proved in the lecture that the stochastic integral process of a similar predictable process with respect to a martingale is martingale.

- (c) Let τ be a stopping time with respect to \mathbb{F} . Show that the stopped process $f(X)^\tau = (f(X_k)^\tau)_{k=0,1,\dots,T}$ defined by $f(X_k)^\tau = f(X_{k \wedge \tau})$ is a submartingale with respect to \mathbb{F} .

Hint: Try to express the stopped process as an appropriate stochastic integral process.

- (d) Let $Y = (Y_k)_{k=0,1,\dots,T}$ be an adapted, integrable process. Show that Y is a martingale if and only if $E[Y_{k+1} | \mathcal{F}_k] = Y_k$ P -a.s. for $k = 0, 1, \dots, T-1$.

Exercise 3.2 Let $(\tilde{S}^0, \tilde{S}^1)$ be a market modeled by a *binomial model*. More precisely, let the undiscounted price processes of the assets in our market be defined by

$$\begin{aligned} \tilde{S}_k^0 &= (1+r)^k & \text{for } k = 0, 1, \dots, T, \\ \frac{\tilde{S}_{k+1}^1}{\tilde{S}_k^1} &= Y_{k+1} & \text{for } k = 0, 1, \dots, T-1, \end{aligned}$$

where the Y_k are i.i.d. random variables taking values $1+u$ with probability $p \in (0, 1)$ and $1+d$ with probability $1-p$. Assume furthermore that $u > d > -1$ and $r > -1$.

- (a) Suppose that $r \leq d$. Show that in this case the market $(\tilde{S}^0, \tilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity*.

- (b) Suppose that $r \geq u$. Show that also in this case the market $(\tilde{S}^0, \tilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity*.

Exercise 3.3 Let $(\tilde{S}^0, \tilde{S}^1)$ be a *trinomial model*. This is like a binomial model a special case of a *multinomial model*, and the distribution of Y_k under P is given by

$$Y_k = \begin{cases} 1+d & \text{with probability } p_1 \\ 1+m & \text{with probability } p_2 \\ 1+u & \text{with probability } p_3 \end{cases}$$

where $p_1, p_2, p_3 > 0$, $p_1 + p_2 + p_3 = 1$ and $-1 < d < m < u$. Here d , m and u are mnemonics for *down*, *middle* and *up*.

- (a) Assume that $d = -0.5$, $m = 0$, $u = 0.25$ and $r = 0$. For $T = 1$, consider an arbitrary self-financing strategy $\varphi \hat{=} (V_0, \theta)$. Show that if the total gain $G_1(\theta)$ at time $T = 1$ is nonnegative P -a.s., then

$$P[G_1(\theta) = 0] = 1.$$

What does this property imply?

- (b) Show that S^1 is arbitrage-free by constructing an *equivalent martingale measure* (EMM) for S^1 .

Hint: A probability measure Q equivalent to P on \mathcal{F}_1 can be uniquely described by a probability vector $(q_1, q_2, q_3) \in (0, 1)^3$, where $q_k = Q[Y_1 = 1 + y_k]$, $k = 1, 2, 3$, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$. (A probability vector in \mathbb{R}^n , $n \in \mathbb{N}$ is a nonnegative vector in \mathbb{R}^n whose coordinates sum up to 1.)

- (c) Assume now that $d = -0.01$, $m = 0.01$, $u = 0.03$ and $r = 0.01$. For $T = 2$, give a parametrization of all EMMs for S^1 .

Hint: A probability measure Q equivalent to P on \mathcal{F}_2 can be uniquely described by four probability vectors (q_1, q_2, q_3) , $(q_{j,1}, q_{j,2}, q_{j,3}) \in (0, 1)^3$, $j = 1, 2, 3$, where $q_j = Q[Y_1 = 1 + y_j]$ and $q_{j,k} = Q[Y_2 = 1 + y_k | Y_1 = 1 + y_j]$, $j, k = 1, 2, 3$, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$.