Coordinator
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Mathematical Foundations for Finance

Exercise sheet 4

Please upload your solutions until Wednesday, 20/10/2021, 12:00 using the link on the course website.

Exercise 4.1 Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. Recall that a *stopping time* is a random variable $\tau : \Omega \to \{0, 1, \dots, T\}$ with the property that

$$\{\tau \le k\} \in \mathcal{F}_k$$

for $k = 0, 1, \dots, T$. Recall also the convention that $\inf \emptyset = +\infty$. If $X = (X_k)_{k=0,1,\dots,T}$ is an $\mathbb{F}$-adapted process and $B \in \mathcal{B}(\mathbb{R})$ a Borel set, then

$$\tau_{X,B} := \inf\{k \in \{0, 1, ..., T\} : X_k \in B\}$$

is called the *first hitting time* of X on B.

(a) Show that $\tau_{X,B} \wedge T$ is a stopping time.

(b) Let τ be any stopping time. Show that there exist an adapted process X and a set $B \in \mathcal{B}(\mathbb{R})$ such that $\tau = \tau_{X,B}$. In other words, show that (up to truncating at T) every (first) hitting time of some adapted process X on some $B \in \mathcal{B}(\mathbb{R})$ is a stopping time and vice versa.
Hint: Try to construct such a process explicitly. It will depend on τ.

Exercise 4.2 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *binomial model* and assume that $T = 1$, $u > r > 0$ and $-1 < d < 0$. For $\widetilde{K} > 0$, define the functions $C(\bullet, \widetilde{K})$ and $P(\bullet, \widetilde{K}) : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$C(x, \widetilde{K}) := (x - \widetilde{K})^+ := \max(0, x - \widetilde{K}) \quad \text{and} \quad P(x, \widetilde{K}) := (\widetilde{K} - x)^+ := \max(0, \widetilde{K} - x).$$

In financial terms, $C(\bullet, \widetilde{K})$ is the payoff function of a *European call option with strike* $\widetilde{K}$, and $P(\bullet, \widetilde{K})$ is the payoff function of a *European put option with strike* $\widetilde{K}$.

(a) Construct a self-financing strategy $\varphi^{C(\widetilde{K})} \mathrel{\widehat{=}} (V_0^{C(\widetilde{K})}, \vartheta^{C(\widetilde{K})})$ such that

$$V_1(\varphi^{C(\widetilde{K})}) = \frac{C(\widetilde{S}_1^1, \widetilde{K})}{1 + r} \quad P\text{-a.s.}$$

Hint: The exercise reduces to solving two linear equations.

(b) Construct a self-financing strategy $\varphi^{P(\widetilde{K})} \mathrel{\widehat{=}} (V_0^{P(\widetilde{K})}, \vartheta^{P(\widetilde{K})})$ such that

$$V_1(\varphi^{P(\widetilde{K})}) = \frac{P(\widetilde{S}_1^1, \widetilde{K})}{1 + r} \quad P\text{-a.s.}$$

Hint: The exercise reduces to solving two linear equations.

(c) Prove the *put-call parity*

$$V_0^{P(\widetilde{K})} + S_0^1 = V_0^{C(\widetilde{K})} + \frac{\widetilde{K}}{1 + r}. \qquad (*)$$

Give an economic interpretation of $(*)$.

(d) Compute $\lim_{\widetilde{K}\to\infty} V_0^{C(\widetilde{K})}$, $\lim_{\widetilde{K}\to 0} V_0^{C(\widetilde{K})}$, $\lim_{\widetilde{K}\to\infty} V_0^{P(\widetilde{K})}$ and $\lim_{\widetilde{K}\to 0} V_0^{P(\widetilde{K})}$. Can you guess the result before doing the computations?

(e) *(Bonus)* Writing P_0, C_0, S_0 and B_0 for the initial price of the put and call with strike $\tilde{K}$, as well as the underlying stock and bond respectively, the put-call parity formula can be rewritten as

$$P_0 - C_0 = B_0 K - S_0,$$

where $K = \tilde{K}/B_0$ is the discounted strike price. This is the equation of a line. Using the programming language of your choice, verify the put-call parity formula on historical prices. To do this, you are asked to

- plot $P_0 - C_0$ versus K, where $t = 0$ corresponds to 23 October 2017 qnd $t = T$ is 17 November 2017, and the underlying asset is the $S\&P500$ index. You can take the price of the calls and puts to be the last traded price on the day (as opposed to bid or ask price). You can find all data needed on yahoo finance.
- perform a linear regression of the response variable $P_0 - C_0$ against the predictor K. What are the obtained coefficients of the regression? Perform a goodness of fit analysis to judge the quality of your fitted model.

Exercise 4.3 Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ with time horizon $T = 1$ consisting of a bank account and one stock defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that $\widetilde{S}_0^0 = \widetilde{S}_0^1 = 1$ and $\widetilde{S}_1^1 = e^Y$, where $Y \sim \mathcal{N}(0,1)$ under P. Finally, assume that $\widetilde{S}_1^0 = e^r$ for a deterministic $r \in (0, 1/2)$ and consider the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 := \mathcal{F}$.

(a) Consider the map $Q : \mathcal{F} \to \mathbb{R}$ given by $Q[A] := E[Z\mathbb{1}_A]$, where

$$Z := \exp\bigg(-\Big(\frac{1}{2} - r\Big)Y - \frac{(\frac{1}{2} - r)^2}{2} \bigg).$$

Show that Q is a probability measure and that it is equivalent to P.
Hint: You can use that for $X \sim \mathcal{N}(\mu, \sigma^2)$, one has $E\left[e^{\alpha X}\right] = \exp(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2)$.

(b) Show that Q is an equivalent martingale measure for S^1, i.e. that S^1 is a martingale under Q.
Hint: In this setting, $E_Q\left[S_1^1\right] = E\left[Z S_1^1\right]$.

(c) Consider again the (undiscounted) payoff $C(\widetilde{S}_1^1, \widetilde{K}) = (\widetilde{S}_1^1 - \widetilde{K})^+$ of a long position in a European call option with strike $\widetilde{K}$. Compute

$$V_0^C := E_Q\left[\frac{C(\widetilde{S}_1^1, \widetilde{K})}{\widetilde{S}_1^0}\right].$$

(d) Consider an enlargement of the market given by $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$, where we set $\widetilde{S}_0^2 := V_0^C$ and $\widetilde{S}_1^2 := C(\widetilde{S}_1^1, \widetilde{K})$. Is this market free of arbitrage?