Mathematical Foundations for Finance

Exercise sheet 4

Please upload your solutions until Wednesday, 20/10/2021, 12:00 using the link on the course website.

Exercise 4.1 Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. Recall that a *stopping time* is a random variable $\tau : \Omega \to \{0, 1, \dots, T\}$ with the property that

$$\{\tau \leq k\} \in \mathcal{F}_k$$

for $k = 0, 1, \dots, T$. Recall also the convention that $\inf \emptyset = +\infty$. If $X = (X_k)_{k=0,1,\dots,T}$ is an $\mathbb{F}$-adapted process and $B \in \mathcal{B}(\mathbb{R})$ a Borel set, then

$$\tau_{X,B} := \inf\{k \in \{0, 1, ..., T\} : X_k \in B\}$$

is called the *first hitting time* of X on B.

(a) Show that $\tau_{X,B} \wedge T$ is a stopping time.

(b) Let τ be any stopping time. Show that there exist an adapted process X and a set $B \in \mathcal{B}(\mathbb{R})$ such that $\tau = \tau_{X,B}$. In other words, show that (up to truncating at T) every (first) hitting time of some adapted process X on some $B \in \mathcal{B}(\mathbb{R})$ is a stopping time and vice versa.
Hint: Try to construct such a process explicitly. It will depend on τ.

Solution 4.1

(a) Fix a $k \in \{0, 1, \dots, T\}$. For any $j \in \{0, 1, \dots, k\}$, X_j is $\mathcal{F}_j$-measurable because X is adapted, which means that $\{X_j \in B\} = \{\omega \in \Omega : X_j(\omega) \in B\} \in \mathcal{F}_j \subset \mathcal{F}_k$. Moreover, by definition of $\tau_{X,B}$, we have

$$\{\tau_{X,B} \leq k\} = \big\{X_j \in B \text{ for some } j \in \{0, 1, \dots, k\}\big\} = \bigcup_{j=0}^{k} \{X_j \in B\} \in \mathcal{F}_k$$

because $\mathcal{F}_k$ as a σ-algebra is closed under countable unions. $\tau_{X,B}$ can, however, attain the value of $+\infty$ and thus does not satisfy our definition of a stopping time. However, since $\tau_{X,B} \wedge T$ can only attain values in $\{0, 1, \dots, T\}$ and

$$\{\tau_{X,B} \wedge T \leq k\} = \begin{cases} \{\tau_{X,B} \leq k\} & \text{for } k < T \\ \Omega & \text{for } k = T, \end{cases}$$

we have that $\tau_{X,B} \wedge T$ indeed is a stopping time.

(b) Given a stopping time τ, we define

$$X_k := \mathbb{1}_{\{\tau > k\}}$$

for $k = 0, 1, \dots, T$ and set $X := (X_k)_{k=0,1,\dots,T}$. Since τ is a stopping time, $\{\tau \leq k\}$ (and therefore $\{\tau > k\} = \{\tau \leq k\}^c$) is in $\mathcal{F}_k$ for every $k = 0, 1, \dots, T$. This implies that X is adapted. Moreover, $X_k(\omega) = 1$ for $\tau(\omega) > k$ and $X_k(\omega) = 0$ for $\tau(\omega) \leq k$ so that

$$\tau(\omega) = \inf\big\{k = 0, \dots, T : X_k(\omega) \in \{0\}\big\}.$$

Therefrom we clearly see that $\tau = \tau_{X,\{0\}}$ for an adapted $X = (X_k)_{k=0,1,\dots,T}$ defined by $X_k = \mathbb{1}_{\{\tau > k\}}$. Note that $\tau \leq T$, so $X_T = 0$ and hence $\tau_{X,\{0\}} \leq T$.

Exercise 4.2 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *binomial model* and assume that $T = 1$, $u > r > 0$ and $-1 < d < 0$. For $\widetilde{K} > 0$, define the functions $C(\bullet, \widetilde{K})$ and $P(\bullet, \widetilde{K}) : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$C(x, \widetilde{K}) := (x - \widetilde{K})^+ := \max(0, x - \widetilde{K}) \quad \text{and} \quad P(x, \widetilde{K}) := (\widetilde{K} - x)^+ := \max(0, \widetilde{K} - x).$$

In financial terms, $C(\bullet, \widetilde{K})$ is the payoff function of a *European call option with strike* $\widetilde{K}$, and $P(\bullet, \widetilde{K})$ is the payoff function of a *European put option with strike* $\widetilde{K}$.

(a) Construct a self-financing strategy $\varphi^{C(\widetilde{K})} \mathrel{\widehat{=}} (V_0^{C(\widetilde{K})}, \vartheta^{C(\widetilde{K})})$ such that

$$V_1(\varphi^{C(\widetilde{K})}) = \frac{C(\widetilde{S}_1^1, \widetilde{K})}{1+r} \quad P\text{-a.s.}$$

Hint: The exercise reduces to solving two linear equations.

(b) Construct a self-financing strategy $\varphi^{P(\widetilde{K})} \mathrel{\widehat{=}} (V_0^{P(\widetilde{K})}, \vartheta^{P(\widetilde{K})})$ such that

$$V_1(\varphi^{P(\widetilde{K})}) = \frac{P(\widetilde{S}_1^1, \widetilde{K})}{1+r} \quad P\text{-a.s.}$$

Hint: The exercise reduces to solving two linear equations.

(c) Prove the *put-call parity*

$$V_0^{P(\widetilde{K})} + S_0^1 = V_0^{C(\widetilde{K})} + \frac{\widetilde{K}}{1+r}. \tag{$*$}$$

Give an economic interpretation of $(*)$.

(d) Compute $\lim_{\widetilde{K}\to\infty} V_0^{C(\widetilde{K})}$, $\lim_{\widetilde{K}\to 0} V_0^{C(\widetilde{K})}$, $\lim_{\widetilde{K}\to\infty} V_0^{P(\widetilde{K})}$ and $\lim_{\widetilde{K}\to 0} V_0^{P(\widetilde{K})}$. Can you guess the result before doing the computations?

(e) *(Bonus)* Writing P_0, C_0, S_0 and B_0 for the initial price of the put and call with strike $\tilde{K}$, as well as the underlying stock and bond respectively, the put-call parity formula can be rewritten as

$$P_0 - C_0 = B_0 K - S_0,$$

where $K = \tilde{K}/B_0$ is the discounted strike price. This is the equation of a line. Using the programming language of your choice, verify the put-call parity formula on historical prices. To do this, you are asked to

- plot $P_0 - C_0$ versus K, where $t = 0$ corresponds to 23 October 2017 qnd $t = T$ is 17 November 2017, and the underlying asset is the $S\&P500$ index. You can take the price of the calls and puts to be the last traded price on the day (as opposed to bid or ask price). You can find all data needed on yahoo finance.
- perform a linear regression of the response variable $P_0 - C_0$ against the predictor K. What are the obtained coefficients of the regression? Perform a goodness of fit analysis to judge the quality of your fitted model.

Solution 4.2

(a) A self-financing strategy $\varphi^{C(\widetilde{K})} \mathrel{\widehat{=}} (V_0^{C(\widetilde{K})}, \vartheta^{C(\widetilde{K})})$ satisfies

$$V_1(\varphi^{C(\widetilde{K})}) = \frac{C(\widetilde{S}^1, \widetilde{K})}{1+r} \quad P\text{-a.s.}$$

if and only if we have

$$V_0^{C(\widetilde{K})} + \vartheta_1^{C(\widetilde{K})} \Delta S_1^1 = \frac{C(\widetilde{S^1}, \widetilde{K})}{1+r} \quad P\text{-a.s.}$$

Since $\vartheta_1^{C(\widetilde{K})}$ is $\mathcal{F}_0$-measurable, hence a constant, and S^1 only takes two values, the latter condition is equivalent to

$$\begin{aligned} V_0^{C(\widetilde{K})} + \vartheta_1^{C(\widetilde{K})} \frac{u-r}{1+r} S_0^1 &= \frac{C((1+u)S_0^1, \widetilde{K})}{1+r} \quad \text{and} \\ V_0^{C(\widetilde{K})} + \vartheta_1^{C(\widetilde{K})} \frac{d-r}{1+r} S_0^1 &= \frac{C((1+d)S_0^1, \widetilde{K})}{1+r}. \end{aligned} \tag{1}$$

Subtracting the two equations, multiplying by $(1+r)$ and dividing by S_0^1 yields

$$\begin{aligned} \vartheta_1^{C(\widetilde{K})}(u-d) &= C(1+u, \widetilde{K}/S_0^1) - C(1+d, \widetilde{K}/S_0^1) \\ \Leftrightarrow \qquad \vartheta_1^{C(\widetilde{K})} &= \frac{C(1+u, \widetilde{K}/S_0^1) - C(1+d, \widetilde{K}/S_0^1)}{u-d}. \end{aligned}$$

Plugging this into (1) yields after rearranging

$$\begin{aligned} V_0^{C(\widetilde{K})} &= \frac{S_0^1}{(1+r)(u-d)} \Big((u-d) C(1+u, \widetilde{K}/S_0^1) \\ &\qquad - (u-r)\big(C(1+u, \widetilde{K}/S_0^1) - C(1+d, \widetilde{K}/S_0^1)\big)\Big) \\ &= \frac{S_0^1}{(1+r)(u-d)} \Big((r-d) C(1+u, \widetilde{K}/S_0^1) + (u-r) C(1+d, \widetilde{K}/S_0^1)\Big) \\ &= S_0^1 \left(\frac{r-d}{u-d} \frac{C(1+u, \widetilde{K}/S_0^1)}{1+r} + \frac{u-r}{u-d} \frac{C(1+d, \widetilde{K}/S_0^1)}{1+r} \right) \\ &= \frac{r-d}{u-d} \frac{C\big(S_0^1(1+u), \widetilde{K}\big)}{1+r} + \frac{u-r}{u-d} \frac{C\big(S_0^1(1+d), \widetilde{K}\big)}{1+r}. \end{aligned}$$

This can also be written as $V_0^{C(\widetilde{K})} = E^*\left[\frac{C(\widetilde{S}_1^1, \widetilde{K})}{1+r}\right]$, where

$$P^*\left[\frac{\widetilde{S}_1^1}{S_0^1} = 1+u\right] = p^* := \frac{r-d}{u-d} \quad \text{and} \quad P^*\left[\frac{\widetilde{S}_1^1}{S_0^1} = 1+d\right] = 1-p^* := \frac{u-r}{u-d}.$$

(b) The same calculations as in (a) yield

$$\begin{aligned} \vartheta_1^{P(\widetilde{K})} &= \frac{P(1+u, \widetilde{K}/S_0^1) - P(1+d, \widetilde{K}/S_0^1)}{u-d}, \\ V_0^{P(\widetilde{K})} &= S_0^1 \left(\frac{r-d}{u-d} \frac{P(1+u, \widetilde{K}/S_0^1)}{(1+r)} + \frac{u-r}{u-d} \frac{P(1+d, \widetilde{K}/S_0^1)}{1+r} \right) = E^*\left[\frac{P(\widetilde{S}_1^1, \widetilde{K})}{1+r}\right]. \end{aligned}$$

(c) For fixed $\widetilde{K} > 0$, by distinguishing the two cases $x \geq \widetilde{K}$ and $x < \widetilde{K}$, we easily get

$$C(x, \widetilde{K}) - P(x, \widetilde{K}) = x - \widetilde{K}.$$

Alternatively, $(\widetilde{K} - x)^+ = (x - \widetilde{K})^-$ gives

$$x - \widetilde{K} = (x - \widetilde{K})^+ - (x - \widetilde{K})^- = C(x, \widetilde{K}) - P(x, \widetilde{K}).$$

1st solution: using a replication argument (primal approach)

Using the above observation and the formulas for $V_0^{C(\widetilde{K})}$ and $V_0^{P(\widetilde{K})}$, we get

$$V_0^{C(\widetilde{K})} - V_0^{P(\widetilde{K})} = \frac{S_0^1}{1+r}\left(\frac{r-d}{u-d}(1+u-\widetilde{K}/S_0^1) + \frac{u-r}{u-d}(1+d-\widetilde{K}/S_0^1)\right)$$
$$= \frac{S_0^1}{1+r}((1-\widetilde{K}/S_0^1)+r) = S_0^1 - \frac{\widetilde{K}}{1+r}. \tag{2}$$

The idea of the replication argument is to construct a portfolio consisting of the risky asset and the bank account such that the terminal value of this portfolio equals the terminal payoff of a portoflio consisting of one long call and one short put option on the rsiky asset and with the same maturities and strikes. No arbitrage then implies that these two portfolios must have the same value at every time step, in particular their initial price must coincide. Using the observation

$$C(x, \widetilde{K}) - P(x, \widetilde{K}) = x - \widetilde{K}.$$

one can easily see that the strategy consisting of

- being long $\tilde{K}$ units of bond with face value 1
- being short one unit of D^1
- being long one unit of C

replicates the terminal payoff of the put option. By no arbitrage, the initial value of the put option must therefore coincide with the initial value of our replicating portfolio which gives the desired put-call parity formula. Note that this replication argument would hold in general in any arbitrage-free market, not just the binomial one studied in this exercise.

2nd solution: using EMM (dual approach)

Alternatively, we can get this by writing

$$V_0^{C(\widetilde{K})} - V_0^{P(\widetilde{K})} = E^*\left[\frac{C(\widetilde{S}_1^1, \widetilde{K}) - P(\widetilde{S}_1^1, \widetilde{K})}{1+r}\right] = E^*\left[\frac{\widetilde{S}_1^1}{1+r}\right] - \frac{\widetilde{K}}{1+r} = S_0^1 - \frac{\widetilde{K}}{1+r}.$$

where E^* denotes the expectation under the unique EMM (the binomial market with $u > r > 0$ is arbitrage-free and complete and hence there exists a unique EMM).

Rearranging yields $(*)$. The economic interpretation of $(*)$ is that buying a stock and a put option with strike $\widetilde{K}$ and maturity T is equivalent to buying a call option with the same strike and the same maturity and a zero-coupon bond with the same maturity and face value $\widetilde{K}$.
Note: Even though we have just proved the put-call parity for a specific model for the market $(\widetilde{S}^0, \widetilde{S}^0)$*, namely the one-period binomial model, no-arbitrage arguments can be used to prove this relationship in a model-free setting.*

(d) For a fixed $x \geq 0$, we clearly have

$$\lim_{\widetilde{K}\to\infty} C(x, \widetilde{K}) = 0 \quad \text{and} \quad \lim_{\widetilde{K}\to 0} C(x, \widetilde{K}) = x.$$

Therefore we have

$$\lim_{\widetilde{K}\to\infty} V_0^{C(\widetilde{K})} = S_0^1\left(\frac{r-d}{u-d}\frac{0}{1+r} + \frac{u-r}{u-d}\frac{0}{1+r}\right) = 0,$$
$$\lim_{\widetilde{K}\to 0} V_0^{C(\widetilde{K})} = S_0^1\left(\frac{r-d}{u-d}\frac{1+u}{1+r} + \frac{u-r}{u-d}\frac{1+d}{1+r}\right) = S_0^1.$$

Using the put-call parity from (2), we get

$$\lim_{\widetilde{K}\to\infty} V_0^{P(\widetilde{K})} = \lim_{\widetilde{K}\to\infty}\left(\frac{\widetilde{K}}{1+r} - S_0^1 + V_0^{C(\widetilde{K})}\right) = +\infty,$$
$$\lim_{\widetilde{K}\to 0} V_0^{P(\widetilde{K})} = \lim_{\widetilde{K}\to 0}\left(\frac{\widetilde{K}}{1+r} - S_0^1 + V_0^{C(\widetilde{K})}\right) = 0.$$

(e) No solution provided.

From $P_0 - C_0 = B_0 K - S_0$, one can directly see that the intercept coefficient should be close to $-S_0$ and the coefficient for K should be close to B_0. For the statistical interpretation of what "close" means and how to analyze the output of a linear regression, as well as how to decide of the quality of the fitted model, you are encouraged to read Chapter 3 of "An Introduction to Statistical Learning" by R. Tibshirani et al.

Exercise 4.3 Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ with time horizon $T = 1$ consisting of a bank account and one stock defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that $\widetilde{S}_0^0 = \widetilde{S}_0^1 = 1$ and $\widetilde{S}_1^1 = e^Y$, where $Y \sim \mathcal{N}(0,1)$ under P. Finally, assume that $\widetilde{S}_1^0 = e^r$ for a deterministic $r \in (0, 1/2)$ and consider the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 := \mathcal{F}$.

(a) Consider the map $Q : \mathcal{F} \to \mathbb{R}$ given by $Q[A] := E[Z\mathbb{1}_A]$, where

$$Z := \exp\left(-\Big(\frac{1}{2} - r\Big)Y - \frac{(\frac{1}{2}-r)^2}{2}\right).$$

Show that Q is a probability measure and that it is equivalent to P.
Hint: You can use that for $X \sim \mathcal{N}(\mu, \sigma^2)$, one has $E\left[e^{\alpha X}\right] = \exp(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2)$.

(b) Show that Q is an equivalent martingale measure for S^1, i.e. that S^1 is a martingale under Q.
Hint: In this setting, $E_Q\left[S_1^1\right] = E\left[Z S_1^1\right]$.

(c) Consider again the (undiscounted) payoff $C(\widetilde{S}_1^1, \widetilde{K}) = (\widetilde{S}_1^1 - \widetilde{K})^+$ of a long position in a European call option with strike $\widetilde{K}$. Compute

$$V_0^C := E_Q\left[\frac{C(\widetilde{S}_1^1, \widetilde{K})}{\widetilde{S}_1^0}\right].$$

(d) Consider an enlargement of the market given by $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$, where we set $\widetilde{S}_0^2 := V_0^C$ and $\widetilde{S}_1^2 := C(\widetilde{S}_1^1, \widetilde{K})$. Is this market free of arbitrage?

Solution 4.3

(a) In order for Q to be a probability measure, we must have that

1. $Q[A] \in [0,1]$ for all $A \in \mathcal{F}$,
2. $Q[\Omega] = 1$,
3. $Q\left[\bigcup_{i=1}^\infty A_i\right] = \sum_{i=1}^\infty Q[A_i]$ for any disjoint family of sets $(A_i)_{i\in\mathbb{N}} \in \mathcal{F}$.

In order for Q to be equivalent to P, we must additionally have that $P[A] = 0 \iff Q[A] = 0$.

For 2, note that setting

$$W := -\Big(\frac{1}{2} - r\Big)Y - \frac{(\frac{1}{2}-r)^2}{2},$$
$$\mu_Z := -\frac{(\frac{1}{2}-r)^2}{2} \qquad \text{and} \qquad \sigma_Z := \frac{1}{2} - r > 0,$$

we can see that $W \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$ under P. Using the hint, we compute

$$E[Z] = E[e^W] = \exp(\mu_Z + \sigma_Z^2/2) = \exp\left(- \frac{(\frac{1}{2}-r)^2}{2} + \frac{(\frac{1}{2}-r)^2}{2} \right) = 1.$$

Next, we have that

$$0 \le Z\mathbb{1}_A \le Z \quad P\text{-a.s. for all } A \in \mathcal{F} \tag{3}$$

where the first inequality follows from the fact that $Z > 0$ P-a.s. (since the function $x \mapsto e^x$ is strictly positive) and the second one from the fact that every $A \in \mathcal{F}$ is a subset of Ω so $\mathbb{1}_A \le \mathbb{1}_\Omega$ and we clearly have that $Z\mathbb{1}_\Omega = Z$. But (3) gives that

$$0 \le E\left[Z\mathbb{1}_A\right] \le E\left[Z\right] = 1,$$

which in turn gives 1.

For 3, consider a family of disjoint sets $(A_i)_{i\in\mathbb{N}} \in \mathcal{F}$. We have that

$$\sum_{i=1}^{\infty} Q\left[A_i\right] = \lim_{n\to\infty} \sum_{i=1}^{n} E\left[Z\mathbb{1}_{A_i}\right] = \lim_{n\to\infty} E\left[\sum_{i=1}^{n} Z\mathbb{1}_{A_i}\right]. \tag{4}$$

However, since the family of sets $(A_i)_{i\in\mathbb{N}}$ is disjoint, the random variable $\sum_{i=1}^n Z\mathbb{1}_{A_i}$ is simply equal to Z on $B = \cup_{i=1}^n A_i$ and to 0 on B^c. So for all $n \in \mathbb{N}$ we have that

$$\left|\sum_{i=1}^{n} Z\mathbb{1}_{A_i}\right| \le Z \quad P\text{-a.s.}$$

Since, $Z \in L^1(P)$, we can use the dominated convergence theorem in (4) to obtain that

$$\sum_{i=1}^{\infty} Q\left[A_i\right] = E\left[\sum_{i=1}^{\infty} Z\mathbb{1}_{A_i}\right] = E\left[Z\mathbb{1}_{\cup_{i\in\mathbb{N}} A_i}\right] = Q\left[\bigcup_{i\in\mathbb{N}} A_i\right],$$

which gives 3.

For the equivalence of Q and P we use that $Z > 0$ P-a.s. which then means that

$$Q[A] = 0 \iff \mathbb{1}_A = 0 \ P\text{-a.s.} \iff E\left[\mathbb{1}_A\right] = 0 \iff P\left[A\right] = 0.$$

(b) In order to prove that Q is an EMM for S^1, we have to show that S^1 is a martingale under Q. Since adaptedness of a process does not depend on a probability measure, we only have to check that S_1^1 is integrable and that $E_Q\left[S_1^1 \,\middle|\, \mathcal{F}_0\right] = E_Q\left[S_1^1\right] = E\left[ZS_1^1\right] = S_0^1$. We can rewrite

$$ZS_1^1 = \exp\left(- \left(\frac{1}{2} - r\right) Y - \frac{(\frac{1}{2}-r)^2}{2} \right) \exp(Y - r)$$

as $ZS_1^1 = e^{\widetilde{Y}}$ with

$$\widetilde{Y} = \left(\frac{1}{2} + r\right) Y - r - \frac{(\frac{1}{2}-r)^2}{2} = \left(\frac{1}{2} + r\right) Y - \frac{(r+\frac{1}{2})^2}{2}.$$

Since $Y \sim \mathcal{N}(0,1)$ under P, we also have that $\widetilde{Y} \sim \mathcal{N}(\mu, \sigma^2)$ under P with

$$\mu = -\frac{(r+\frac{1}{2})^2}{2} \qquad \text{and} \qquad \sigma^2 = (\frac{1}{2} + r)^2.$$

Using the hint, we obtain that

$$E_Q\left[|S_1^1|\right] = E_Q\left[S_1^1\right] = E\left[ZS_1^1\right] = E\left[e^{\widetilde{Y}}\right] = 1 = \widetilde{S}_0^1,$$

giving both the integrability of S_1^1 and the martingale property of S^1 under Q.

(c) Let's start by writing $C(\widetilde{S}_1^1, \widetilde{K}) = (\widetilde{S}_1^1 - \widetilde{K})\mathbb{1}_{\{\widetilde{S}_1^1 \geq \widetilde{K}\}} = (e^Y - \widetilde{K})\mathbb{1}_{\{Y \geq \log \widetilde{K}\}}$. Then

$$\begin{aligned}
E_Q\left[\frac{C(\widetilde{S}_1^1, \widetilde{K})}{\widetilde{S}_1^0}\right] &= e^{-r} E_Q[(e^Y - \widetilde{K})1_{\{Y \geq \log \widetilde{K}\}}] \\
&= e^{-r} E_P[Z(e^Y - \widetilde{K})1_{\{Y \geq \log \widetilde{K}\}}] \\
&= e^{-r} \int_{\log \widetilde{K}}^{\infty} (e^y - \widetilde{K}) \exp\left(-\left(\frac{1}{2} - r\right) y - \frac{\left(\frac{1}{2} - r\right)^2}{2}\right) \phi(y) dy \\
&= e^{-r}(I_1 + I_2),
\end{aligned}$$

where ϕ denotes the density of $\mathcal{N}(0,1)$. These integrals are calculated by completing squares as follows:

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}} \int_{\log \widetilde{K}}^{\infty} \exp\left(\left(\frac{1}{2} + r\right) y - \frac{\left(\frac{1}{2} - r\right)^2}{2} - \frac{y^2}{2}\right) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{\log \widetilde{K}}^{\infty} \exp\left(-\frac{\left(y - \left(\frac{1}{2} + r\right)\right)^2}{2} + r\right) dy \\
&= \frac{e^r}{\sqrt{2\pi}} \int_{\log \widetilde{K} - \left(\frac{1}{2} + r\right)}^{\infty} e^{-\frac{x^2}{2}} dx \\
&= e^r \left(1 - \Phi\left(\log \widetilde{K} - \left(\frac{1}{2} + r\right)\right)\right) = e^r \Phi\left(-\log \widetilde{K} + \left(\frac{1}{2} + r\right)\right).
\end{aligned}$$

In a similar fashion,

$$\begin{aligned}
I_2 &= -\frac{\widetilde{K}}{\sqrt{2\pi}} \int_{\log \widetilde{K}}^{\infty} \exp\left(-\left(\frac{1}{2} - r\right) y - \frac{\left(\frac{1}{2} - r\right)^2}{2} - \frac{y^2}{2}\right) dy \\
&= -\frac{\widetilde{K}}{\sqrt{2\pi}} \int_{\log \widetilde{K}}^{\infty} \exp\left(-\frac{\left(y - \left(r - \frac{1}{2}\right)\right)^2}{2}\right) dy \\
&= -\widetilde{K} \Phi\left(-\log \widetilde{K} + \left(r - \frac{1}{2}\right)\right).
\end{aligned}$$

Returning to the original expression yields

$$E_Q\left[\frac{C(\widetilde{S}_1^1, \widetilde{K})}{\widetilde{S}_1^0}\right] = \Phi\left(-\log \widetilde{K} + \left(r + \frac{1}{2}\right)\right) - \widetilde{K} e^{-r} \Phi\left(-\log \widetilde{K} + \left(r - \frac{1}{2}\right)\right).$$

(d) Yes, the extended market remains arbitrage free, since by construction the measure Q is an equivalent martingale measure not only for S^1 but also for and S^2. Lemma II.1.2 in the lecture notes thus gives us that the desired result.