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Mathematical Foundations for Finance

Exercise sheet 5

Please upload your solutions until Wednesday, 27/10/2021, 12:00 using the link on the course website.

Exercise 5.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_k)_{k=0,\ldots,T}$ in finite discrete time. The goal of this exercise is to explicitly construct an equivalent martingale measure Q. The exercise closely follows pages 40 to 44 from the lecture notes.

(a) Prove that the existence of a measure $Q \approx P$ on $\mathcal{F}$ is equivalent to the existence of a pair (Z_0, D) satisfying all of the following properties.

- (i) $Z_0 > 0 \quad P$-as,
- (ii) $E_P[Z_0] = 1$,
- (iii) $D = (D_k)_{k=0,\ldots,T}$ adapted and strictly positive stochastic process,
- (iv) $E_P[D_k \,|\, \mathcal{F}_{k-1}] = 1$ for all k.

(b) Let $(\tilde{S}^0, \tilde{S}^1)$ be a financial market satisfying (i)-(iv) above with iid returns, i.e suppose the price dynamics are given by

$$\tilde{S}^1_k = S^1_0 \prod_{j=1}^{k} Y_j; \qquad \tilde{S}^0_k = (1+r)^k$$

for iid random variables Y_j for $j \in \{1, \ldots, T\}$. Assume that the filtration is generated by the returns process Y and that $\mathcal{F}_0$ is P-trivial.

Our goal is to construct an equivalent measure Q by explicitly deriving a pair (Z_0, D) statisfying

- $Z_0 > 0 \quad P$-as.
- $E_P[Z_0] = 1$
- $D = (D_k)_{k=0,\ldots,T}$ adapted and strictly positive stochastic process
- $E_P[D_k \,|\, \mathcal{F}_{k-1}] = 1$

Moreover, we want Q to be a martingale measure and thus also ask for

- $E_Q\left[\frac{S^1_k}{S^1_{k-1}} \,\middle|\, \mathcal{F}_{k-1}\right] = E_Q\left[\frac{\tilde{S}^1_k/\tilde{S}^0_k}{\tilde{S}^1_{k-1}/\tilde{S}^0_k} \,\middle|\, \mathcal{F}_{k-1}\right] = E_Q\left[\frac{Y_k}{1+r} \,\middle|\, \mathcal{F}_{k-1}\right] = E_P\left[\frac{D_k Y_k}{1+r} \,\middle|\, \mathcal{F}_{k-1}\right] = 1$

[Note that we have used Bayes theorem to relate the conditional expectation under Q to the one under P; see top of page 42 in your notes]

To keep things simple, let's take $Z_0 = 1$ which clearly satisfies the required properties. Also assume that D_k is independent of $\mathcal{F}_{k-1}$ (like Y_k) and moreover that $D_k = g_k(Y_k)$ for some Borel-measurable function g_k. Derive conditions on g_k that make the measure Q defined by the Radon-Nykodym derivative

$$\frac{dQ}{dP} = Z_0 \prod_{j=1}^{T} D_j = Z_0 \prod_{j=1}^{T} g_j(Y_j)$$

become an equivalent martingale measure.

For simplicity from now on we will choose $g_k = g$ for all k where g satisfies the properties derived in the previous question. This is an admissible choice since the returns Y_k are assumed to be i.i.d. under P.

(c) From now on, suppose that we have i.i.d lognormal returns, i.e $Y_k = \exp(\sigma U_k + b)$ with random variables $U_1, \dots, U_T$ i.i.d. $\sim \mathcal{N}(0,1)$. Instead of $D_k = g_k(Y_k) = g(Y_k)$, we here try (equivalently) to find a function f such that $D_k = f(U_k)$. Take $f(x) := \exp(\alpha x + \beta)$. Derive conditions on α and β such that the measure Q defined by the Radon-Nykodym derivative

$$\frac{dQ}{dP} = Z_0 \prod_{j=1}^{T} D_j = Z_0 \prod_{j=1}^{T} f(U_j)$$

becomes an equivalent martingale measure.

Exercise 5.2 Let $(Y_t)_{0 \le t \le T}$ be a given integrable adapted discrete-time process. Define an adapted process $(U_t)_{0 \le t \le T}$ by the recursion

$$\begin{aligned} U_T &= Y_T \\ U_t &= \max\left(Y_t, E[U_{t+1}|\mathcal{F}_t]\right) \quad \text{for } 0 \le t \le T-1 \end{aligned}$$

The process $(U_t)_{0 \le t \le T}$ is called the Snell envelope of $(Y_t)_{0 \le t \le T}$. For simplicity, we suppose in this exercise that $\mathcal{F}_0$ is the trivial σ-algebra.

(a) Show that the Snell envelope of a process is the smallest supermartingale dominating that process.

(b) Show that if Y is a supermartingale then $U_t = Y_t$ for all t, and if Y is a submartingale, then $U_t = E[Y_T|\mathcal{F}_t]$.

(c) Let τ be any stopping time taking values in $\{0, ..., T\}$. Show that the process $(U_{t \wedge \tau})_{0 \le t \le T}$ is a supermartingale.

Define the random time τ^* by

$$\tau^* = \min\{t \in \{0, ..., T\} \text{ such that } U_t = Y_t\}$$

(d) Show that τ^* is a stopping time. Furthermore, show that the process $(U_{t \wedge \tau^*})_{0 \le t \le T}$ is a martingale and, in particular, $U_0 = E[Y_{\tau^*}]$

(e) Show that $U_0 = \sup\{E[Y_\tau] : \text{stopping times } 0 \le \tau \le T\}$

(f) Conclude that τ^* is an optimal stopping time, i.e. a solution to the problem of finding a stopping time $\tau \le T$ that achieves the supremum in $\sup_{\tau \le T} E[Y_\tau]$.

(g) Give a financial example where this result could be used.

Exercise 5.3 *This is an optional exercise. You are highly encouraged to solved it, but the results of this exercise are not part of the exam material.* This exercise guides you through an alternative proof of the "hard" direction of the First Fundamental Theorem of Asset Pricing (also known as Dalang-Morton-Willinger Theorem). In this exercise we will focus on the basic one-period model, i.e we suppose that $T = 1$. The proof for the multi-period case is very similar but is a little more difficult because of some technicalities involving measurability. For simplicity, we also assume that $\mathcal{F}_0$ is (P-) trivial, so θ predictable means $\theta \in \mathbb{R}^d$. Moreover we also suppose that there exists a numéraire asset.

Let $(\tilde{S}_0^0, \tilde{S}_0^1)$ (respectively $(\tilde{S}_1^0, \tilde{S}_1^1)$) denote the vector of initial undiscounted prices (respectively terminal undiscounted prices), and let $(1, S_t^1)_{t \in \{0,1\}}$ be the discounted (with respect to the numéraire asset $\tilde{S}^0$) price process.

(a) Define a *pricing kernel* (also called *stochastic discount factor* or *state price density*) as a strictly positive random variable ρ satisfying

$$\tilde{S}_0^1 = \mathbb{E}_P \left[\rho \tilde{S}_1^1\right]$$

where P is the objective (or historical or statistical) measure of our filtered probability space $(\Omega, \mathcal{F}, P)$. When the market has a numéraire, we can characterize pricing kernels in terms of the discounted prices $(1, S^1)$: the pricing kernel ρ is a positive random variable $\rho > 0$ in $L^\infty(P)$ satisfying

$$E_P \left[\rho \Delta S_1^1\right] = 0$$

Show that when the market has a numéraire, the notion of a pricing kernel and that of an EMM are essentially the same. More precisely, show that the measure Q defined by

$$\frac{dQ}{dP} = \frac{\rho}{E_P[\rho]}$$

gives an EMM.

Since we suppose the existence of a numéraire, by a general result, the market is arbitrage free iff there is no arbitrage of the first kind, i.e. no arbitrage as defined in this lecture[1]. Moreover by question (a) the existence of a pricing kernel is equivalent to the existence of an EMM. We thus have to show that no arbitrage (of first kind) implies the existence of a pricing kernel ρ.

(b) Consider the function $F \colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ defined by

$$F(\theta) = \mathbb{E}_P \left[e^{-\theta \cdot \Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2}\right]$$

Show that F is finite valued and smooth (C^1).

(c) Suppose that there exists a minimiser θ^* of F. Construct a pricing kernel ρ and show that the corresponding EMM Q has a bounded Radon Nykodym derivative, i.e. $\frac{dQ}{dP} \in L^\infty$.

(d) In this question we show that the no arbitrage (of first kind) assumption implies the existence of a minimiser θ^* of F.

- Let $(\theta_k)_k$ be a minimizing sequence, i.e a sequence that satisfies

$$\lim_{k \to \infty} F(\theta_k) = \inf_{\theta \in \mathbb{R}^d} F(\theta)$$

  Suppose that $(\theta_k)_k$ is bounded. Show that in this case F admits a minimiser θ^*.

It remains to show that no arbitrage (of first kind) implies the existence of a bounded minimising sequence $(\theta_k)_k$.

Let $\mathcal{U} = \{\theta \in \mathbb{R}^d : \theta \cdot \Delta S_1^1 = 0 \ P\text{-}a.s\} \subseteq \mathbb{R}^d$ and $\mathcal{V} = \mathcal{U}^\perp$ the orthogonal complement of $\mathcal{U}$.

- Show that if $u \in \mathcal{U}$ and $v \in \mathcal{V}$ then $F(u + v) = F(v)$

Choose a minimising sequence $(\theta_k)_k$. By the previous result we can assume without loss of generality that $\theta_k \in \mathcal{V}$ for all k (otherwise we can construct a minimising sequence valued in $\mathcal{V}$ by projecting the original sequence $(\theta_k)_k$ on $\mathcal{V}$. The obtained projected sequence is still a minimising sequence since the projection does not change the value of the function $F(\cdot)$ by the previous question). Assume by contradiction that $(\theta_k)_k$ is unbounded, i.e after passing to a subsequence (again we continue to denote it by $(\theta_k)_k$), $||\theta_k|| \to \infty$. The goal of the next questions is to use the No Arbitrage assumption to get a contradiction.

[1] There are several types of arbitrages, but in this course we only study arbitrages of the first kind because we always assume the existence of a numéraire

- Since $(\theta_k)_k$ is unbounded, we can pass to a subsequence such that $||\theta_k|| \to \infty$. Define $\hat{\theta}_k = \frac{\theta_k}{||\theta_k||}$.
  Show that $\hat{\theta}_k \in \mathcal{V}$ and $||\hat{\theta}_k|| = 1$.

By Bolzano-Weierstrass Theorem, the bounded sequence $(\hat{\theta}_k)_k$ admits a converging subsequence. Let $\hat{\theta}_k$ denote this converging subsequence and let $\hat{\theta}$ be the limit of $\hat{\theta}_k$.

- Show that $\mathcal{V}$ is a closed set and conclude that $\hat{\theta} \in \mathcal{V}$. Show also that $\hat{\theta} \in \mathcal{V}$ and has unit norm.
- Show that the sequence $(F(\theta_k))_k$ is bounded.
- By showing that
  $$F(\theta_k) = E_P\left[\left(e^{-\hat{\theta}_k \cdot \Delta S_1^1}\right)^{||\theta_k||} e^{-\frac{||\Delta S_1^1||^2}{2}}\right]$$
  conclude that we must have $\hat{\theta} \cdot \Delta S_1^1 \geq 0$ a.s.
- By using the no arbitrage assumption find a contradiction. Conclude that $(\theta_k)_k$ is bounded.