Mathematical Foundations for Finance

Exercise sheet 5

Please upload your solutions until Wednesday, 27/10/2021, 12:00 using the link on the course website.

Exercise 5.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_k)_{k=0,\dots,T}$ in finite discrete time. The goal of this exercise is to explicitly construct an equivalent martingale measure Q. The exercise closely follows pages 40 to 44 from the lecture notes.

(a) Prove that the existence of a measure $Q \approx P$ on $\mathcal{F}$ is equivalent to the existence of a pair (Z_0, D) satisfying all of the following properties.

(i) $Z_0 > 0 \quad P$-as,

(ii) $E_P[Z_0] = 1$,

(iii) $D = (D_k)_{k=0,\dots,T}$ adapted and strictly positive stochastic process,

(iv) $E_P[D_k \,|\, \mathcal{F}_{k-1}] = 1$ for all k.

(b) Let $(\tilde{S}^0, \tilde{S}^1)$ be a financial market satisfying (i)-(iv) above with iid returns, i.e suppose the price dynamics are given by

$$\tilde{S}^1_k = S^1_0 \prod_{j=1}^{k} Y_j; \qquad \tilde{S}^0_k = (1+r)^k$$

for iid random variables Y_j for $j \in \{1, \dots, T\}$. Assume that the filtration is generated by the returns process Y and that $\mathcal{F}_0$ is P-trivial.

Our goal is to construct an equivalent measure Q by explicitly deriving a pair (Z_0, D) statisfying

- $Z_0 > 0 \quad P$-as.
- $E_P[Z_0] = 1$
- $D = (D_k)_{k=0,\dots,T}$ adapted and strictly positive stochastic process
- $E_P[D_k \,|\, \mathcal{F}_{k-1}] = 1$

Moreover, we want Q to be a martingale measure and thus also ask for

- $E_Q\left[\frac{S^1_k}{S^1_{k-1}} \,\middle|\, \mathcal{F}_{k-1}\right] = E_Q\left[\frac{\tilde{S}^1_k/\tilde{S}^0_k}{\tilde{S}^1_{k-1}/\tilde{S}^0_k} \,\middle|\, \mathcal{F}_{k-1}\right] = E_Q\left[\frac{Y_k}{1+r} \,\middle|\, \mathcal{F}_{k-1}\right] = E_P\left[\frac{D_k Y_k}{1+r} \,\middle|\, \mathcal{F}_{k-1}\right] = 1$

[Note that we have used Bayes theorem to relate the conditional expectation under Q to the one under P; see top of page 42 in your notes]

To keep things simple, let's take $Z_0 = 1$ which clearly satisfies the required properties. Also assume that D_k is independent of $\mathcal{F}_{k-1}$ (like Y_k) and moreover that $D_k = g_k(Y_k)$ for some Borel-measurable function g_k. Derive conditions on g_k that make the measure Q defined by the Radon-Nykodym derivative

$$\frac{dQ}{dP} = Z_0 \prod_{j=1}^{T} D_j = Z_0 \prod_{j=1}^{T} g_j(Y_j)$$

become an equivalent martingale measure.

For simplicity from now on we will choose $g_k = g$ for all k where g satisfies the properties derived in the previous question. This is an admissible choice since the returns Y_k are assumed to be i.i.d. under P.

(c) From now on, suppose that we have i.i.d lognormal returns, i.e $Y_k = \exp(\sigma U_k + b)$ with random variables $U_1, \dots, U_T$ i.i.d. $\sim \mathcal{N}(0,1)$. Instead of $D_k = g_k(Y_k) = g(Y_k)$, we here try (equivalently) to find a function f such that $D_k = f(U_k)$. Take $f(x) := \exp(\alpha x + \beta)$. Derive conditions on α and β such that the measure Q defined by the Radon-Nykodym derivative

$$\frac{dQ}{dP} = Z_0 \prod_{j=1}^{T} D_j = Z_0 \prod_{j=1}^{T} f(U_j)$$

becomes an equivalent martingale measure.

Solution 5.1

(a) We have to show an if and only if statement so there are two implications to prove.

First suppose that we are given $Q \approx P$ on $\mathcal{F}$. Since $Q \approx P$, the Radon-Nikodym derivative $\frac{dQ}{dP}$ of Q with respect to P must be stricly positive (as otherwise there could exist $A \in \mathcal{F}$ such that $Q[A] = E_P\left[\frac{dQ}{dP}\mathbb{1}_A\right] = 0$ but $P[A] \neq 0$). Let us introduce the density process Z of Q with respect to P as follows:

$$Z_k := E_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_k\right] \qquad k = 0, \dots, T$$

Because $\frac{dQ}{dP} > 0$, we have $Z_k > 0$ P-as for all $k = 0, \dots, T$. We can thus define

$$D_k := \frac{Z_k}{Z_{k-1}} \qquad k = 1, \dots, T$$

Remains to show that the pair (Z_0, D) satisfies the required properties.

- $Z_0 > 0 \quad P$-as:
  Indeed we have already seen that the process Z is striclty positive.
- $E_P[Z_0] = 1$:
  Using the tower property,

  $$E_P[Z_0] = E_P\left[E_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_0\right]\right] = E_P\left[\frac{dQ}{dP}\right] = E_P\left[\frac{dQ}{dP}\mathbb{1}_\Omega\right] = Q[\Omega] = 1$$

  where in the last equalities we have used Bayes Theorem together with the fact that Q is a probability measure.
  Alternative solution: It is easy to see that Z is a martingale (good exercise!) under the measure P. Therefore is is constant in expectation and thus

  $$E_P[Z_0] = E_P[Z_T] = E_P[Z_T \mathbb{1}_\Omega] = Q[\Omega] = 1$$

- $D = (D_k)_{k=0,\dots,T}$ adapted and strictly positive stochastic process:
  Adaptedness and striclty positivity follow from the same properties of Z.

- $E_P[D_k \,|\, \mathcal{F}_{k-1}] = 1$

$$\begin{aligned} E_P[D_k \,|\, \mathcal{F}_{k-1}] &= E_P\left[\frac{Z_k}{Z_{k-1}} \,\middle|\, \mathcal{F}_{k-1}\right] \\ &= \frac{1}{Z_{k-1}} E_P[Z_k \,|\, \mathcal{F}_{k-1}] \\ &= \frac{1}{Z_{k-1}} E_P\left[E_P\left[\frac{dQ}{dP} \,\middle|\, \mathcal{F}_k\right] \,\middle|\, \mathcal{F}_{k-1}\right] \\ &= \frac{1}{Z_{k-1}} E_P\left[\frac{dQ}{dP} \,\middle|\, \mathcal{F}_{k-1}\right] \\ &= \frac{Z_{k-1}}{Z_{k-1}} = 1 \end{aligned}$$

where we have used the definition of Z_k and the tower property.

Conversely suppose the we are given a pair (Z_0, D) satisfying the above conditions. We can then define the whole process Z via

$$Z_k := Z_0 \prod_{j=1}^{k} D_j \qquad k = 0, \dots, T$$

We claim that the measure Q defined by the Radon-Nikodym derivative

$$\frac{dQ}{dP} = Z_T$$

is a probability measure equivalent to P on $\mathcal{F}$. Indeed, Q is a probability measure since

- $Q[\Omega] = E_P\left[\frac{dQ}{dP}\mathbb{1}_\Omega\right] = E_P[Z_T]$
  We could conclude if we had $E_P[Z_T] = 1$. This will follow if we manage to show that the process Z defined above is a martingale. Indeed, then $E_P[Z_T] = E_P[Z_0]$ but $E_P[Z_0] = 1$ by assumption. So it remains to prove that Z is indeed a martingale. Adaptedness follows from the adapteness of D. Integrability is a bit tricky to prove. First note that since $Z_k > 0$, it is enough to show $E_P[Z_k] < \infty$. In order to prove this, we will proceed by induction.
  First consider the case $k = 0$: $E_P[Z_0] = 1 < \infty$ by assumption.
  Next consider the case $k = 1$:

$$\begin{aligned} E_P[Z_1] = E_P[Z_0 D_1] &= E_P[E_P[Z_0 D_1 \,|\, \mathcal{F}_0]] \\ &= E_P[Z_0 E_P[D_1 \,|\, \mathcal{F}_0]] \\ &= E_P[Z_0] = 1 < \infty \end{aligned}$$

  where we have used (in order) the tower property, the adaptedness of Z, the assumption $E_P[D_k \,|\, \mathcal{F}_{k-1}] = 1$, and finally the assumption $E_P[Z_0] = 1$.
  Next we suppose that $E_P[Z_k] < \infty$ and we show that $E_P[Z_{k+1}] < \infty$

$$\begin{aligned} E_P[Z_{k+1}] = E_P[Z_k D_{k+1}] &= E_P[E_P[Z_k D_{k+1} \,|\, \mathcal{F}_k]] \\ &= E_P[Z_k E_P[D_{k+1} \,|\, \mathcal{F}_k]] \\ &= E_P[Z_k] < \infty \end{aligned}$$

where the same arguments have been used as above and the last inequality comes from the induction hypotheses. This shows that Z is indeed integrable. Remains to show the martingale property:

$$\begin{aligned} E_P\left[Z_{k+1} \,|\, \mathcal{F}_k\right] &= E_P\left[Z_k D_{k+1} \,|\, \mathcal{F}_k\right] \\ &= Z_k E_P\left[D_{k+1} \,|\, \mathcal{F}_k\right] \\ &= D_k \end{aligned}$$

where we used (in order) the adaptedness of Z and the assumption $E_P\left[D_k \,|\, \mathcal{F}_{k-1}\right] = 1$. This concludes the prove that Z is a true martingale (which in turn implies $Q[\Omega] = 1$ as explained above).

- $Q[A] \in [0,1] \quad \forall A \in \mathcal{F}$
  We clearly have
  $$0 \leq \frac{dQ}{dP}\mathbb{1}_A \leq \frac{dQ}{dP}\mathbb{1}_\Omega$$
  taking expectations under P in the above equation gives:
  $$0 \leq Q[A] \leq Q[\Omega] = 1$$
- For the sigma-additivity, consider a family of disjoint sets $(A_i)_{i\in\mathbb{N}} \in \mathcal{F}$. We have that
  $$\sum_{i=1}^{\infty} Q\left[A_i\right] = \lim_{n\to\infty} \sum_{i=1}^{n} E_P\left[Z_T \mathbb{1}_{A_i}\right] = \lim_{n\to\infty} E_P\left[\sum_{i=1}^{n} Z_T \mathbb{1}_{A_i}\right]. \tag{1}$$
  However, since the family of sets $(A_i)_{i\in\mathbb{N}}$ is disjoint, the random variable $\sum_{i=1}^{n} Z_T \mathbb{1}_{A_i}$ is simply equal to Z_T on $B = \cup_{i=1}^{n} A_i$ and to 0 on B^c. So for all $n \in \mathbb{N}$ we have that
  $$\left|\sum_{i=1}^{n} Z_T \mathbb{1}_{A_i}\right| \leq Z_T \quad P\text{-a.s.}$$
  Since, $Z_T \in L^1(P)$, we can use the dominated convergence theorem in (1) to obtain that
  $$\sum_{i=1}^{\infty} Q\left[A_i\right] = E_P\left[\sum_{i=1}^{\infty} Z_T \mathbb{1}_{A_i}\right] = E_P\left[Z_T \mathbb{1}_{\cup_{i\in\mathbb{N}} A_i}\right] = Q\left[\bigcup_{i\in\mathbb{N}} A_i\right],$$
  which gives the sigma-additivity of the measure Q.
- It remains to show that Q is equivalent to P. But this is clear since $Z_T > 0$ and therefore
  $$Q[A] = 0 \iff E_P\left[Z_T \mathbb{1}_A\right] = 0 \iff Z_T \mathbb{1}_A = 0 \quad P\text{-as} \iff \mathbb{1}_A = 0 \quad P\text{-as} \iff P[A] = 0$$

(b) Adaptedness of $D_k = g_k(Y_k)$ is clear because g_k is supposed to be Borel measurable and Y is clearly adapted to its natural filtration. To ensure strict positivity of D, we will only consider functions g_k that take strictly positive values. Remains to ensure the following properties for all k:

- $E_P\left[D_k \,|\, \mathcal{F}_{k-1}\right] = E_P\left[g_k(Y_k) \,|\, \mathcal{F}_{k-1}\right] = 1$
- $E_P\left[\frac{D_k Y_k}{1+r} \,\middle|\, \mathcal{F}_{k-1}\right] = 1$

Using the independence of (D_k, Y_k) from the sigma-algebra $\mathcal{F}_{k-1}$, the above conditions become

- $E_P\left[D_k\right] = E_P\left[g_k(Y_k)\right] = 1$
- $E_P\left[D_k Y_k\right] = E_P\left[g_k(Y_k) Y_k\right] = 1 + r$

(c) By the previous question, we know that we the parameters of f must be chosen such that

- f is stricly positive –> this is true for any α and β
- $E_P[D_k] = E_P[f(U_k)] = 1$
- $E_P[D_k Y_k] = E_P[f(U_k) Y_k] = 1 + r$

The second condition gives

$$\exp\left(\beta + \frac{1}{2}\alpha^2\right) = 1$$

where we used the moment generating function of the Gaussian random variable $\alpha U_k + \beta$ to compute $E_P[D_k]$. Hence $\beta = -\frac{\alpha^2}{2}$.

To satisfy the third condition, we need (again using the moment generating function of the Gaussian random variable $D_k Y_k = (\alpha + \sigma) U_k + \beta + b)$)

$$\exp\left(b + \beta + \frac{1}{2}(\alpha + \sigma)^2\right) = 1 + r$$

which using that $\beta = -\frac{\alpha^2}{2}$ and taking the log on both sides gives

$$\alpha = \frac{\log(1 + r) - b - \frac{1}{2}\sigma^2}{\sigma}$$

Exercise 5.2 Let $(Y_t)_{0 \leq t \leq T}$ be a given integrable adapted discrete-time process. Define an adapted process $(U_t)_{0 \leq t \leq T}$ by the recursion

$$\begin{aligned} U_T &= Y_T \\ U_t &= \max\left(Y_t, E[U_{t+1} | \mathcal{F}_t]\right) \quad \text{for } 0 \leq t \leq T - 1 \end{aligned}$$

The process $(U_t)_{0 \leq t \leq T}$ is called the Snell envelope of $(Y_t)_{0 \leq t \leq T}$. For simplicity, we suppose in this exercise that $\mathcal{F}_0$ is the trivial σ-algebra.

(a) Show that the Snell envelope of a process is the smallest supermartingale dominating that process.

(b) Show that if Y is a supermartingale then $U_t = Y_t$ for all t, and if Y is a submartingale, then $U_t = E[Y_T | \mathcal{F}_t]$.

(c) Let τ be any stopping time taking values in $\{0, ..., T\}$. Show that the process $(U_{t \wedge \tau})_{0 \leq t \leq T}$ is a supermartingale.

Define the random time τ^* by

$$\tau^* = \min\{t \in \{0, ..., T\} \text{ such that } U_t = Y_t\}$$

(d) Show that τ^* is a stopping time. Furthermore, show that the process $(U_{t \wedge \tau^*})_{0 \leq t \leq T}$ is a martingale and, in particular, $U_0 = E[Y_{\tau^*}]$

(e) Show that $U_0 = \sup\{E[Y_\tau] : \text{stopping times } 0 \leq \tau \leq T\}$

(f) Conclude that τ^* is an optimal stopping time, i.e. a solution to the problem of finding a stopping time $\tau \leq T$ that achieves the supremum in $\sup_{\tau \leq T} E[Y_\tau]$.

(g) Give a financial example where this result could be used.

Solution 5.2

(a) The integrability and adaptedness of U follow from the same properties of Y. Formally one should use backward induction to prove integrability. This is left as an easy but very good exercise. Moreover, by definition, $U_t \geq Z_t$ and $U_t \geq E[U_{t+1}|\mathcal{F}_t]$ for all $0 \leq t \leq T$ hence the Snell envelope U of the process Y is a supermartingale dominating the process Y. Remains to show that U is the smallest such process. Let $V = (V_n)$ be any other supermartingale dominating Y, i.e. $V_n \geq Y_n$ for all n. We have to show that V dominates U as well. We do this by (backwards) induction. First, since $U_T = Y_T$ and V dominates Y, we have $V_T \geq U_T$. Assume inductively that $V_t \geq U_t$. Then

$$\begin{aligned} V_{t-1} &\geq E[V_t|\mathcal{F}_{t-1}] \quad \textit{as } V \textit{ is a supermartingale} \\ &\geq E[U_t|\mathcal{F}_{t-1}] \quad \textit{by the induction hypotheses} \end{aligned}$$

and also $V_{t-1} \geq Y_{t-1}$ as V dominates Y. Combining these two observations and using the definition of Snell enveleppe, we have

$$V_{t-1} \geq U_{t-1}$$

as required.

(b) In both cases we proceed by induction.

First suppose that Y is a supermartingale, and that we have proved $U_{t+1} = Y_{t+1}$ for some $t < T$. Then

$$U_t = \max\left(Y_t, E[U_{t+1}|\mathcal{F}_t]\right) = \max\left(Y_t, E[Y_{t+1}|\mathcal{F}_t]\right) = Y_t$$

where in the second equality we used the induction hypotheses and the last equality holds since Y is a supermartingale by assumption.

Now suppose that Y is a submartingale and that we have proved $U_{t+1} = E[Y_T|\mathcal{F}_{t+1}]$ for some $t < T$. Then

$$U_t = \max\left(Y_t, E[U_{t+1}|\mathcal{F}_t]\right) = \max\left(Y_t, E\left[E[Y_T|\mathcal{F}_{t+1}]|\mathcal{F}_t\right]\right) = E[Y_T|\mathcal{F}_t]$$

where in the second equality we used the induction hypotheses and the last equality holds by the tower property and the assumption that Y is a submartingale.

(c) Note that $U_{(t+1)\wedge\tau} - U_{t\wedge\tau} = \mathbb{1}_{t+1\leq\tau}(U_{t+1} - U_t)$. The supermartingale property of $(U_{t\wedge\tau})_{0\leq t\leq T}$ now immediately follows from the supermartingale property of U

$$\begin{aligned} E\left[U_{(t+1)\wedge\tau} - U_{t\wedge\tau}|\mathcal{F}_t\right] &= E\left[\mathbb{1}_{t+1\leq\tau}(U_{t+1} - U_t)|\mathcal{F}_t\right] \\ &= \mathbb{1}_{t+1\leq\tau} E\left[(U_{t+1} - U_t)|\mathcal{F}_t\right] \\ &\leq 0 \end{aligned}$$

where in the second line we used that $\mathbb{1}_{t+1\leq\tau} = 1 - \mathbb{1}_{\tau\leq t}$ is $\mathcal{F}_t$ measurable and the last inequality holds since U is a supermartingale. Adaptedness and integrability are trivial.

(d) The event

$$\{\tau^* > t\} = \{Y_0 < U_0, ..., Y_t < U_t\}$$

is $\mathcal{F}_t$ measurable since both U and Y are adapted processes, hence τ^* is indeed a stopping time.

Now note that on the event $\{t+1 \leq \tau^*\}$, $U_t = E[U_{t+1}|\mathcal{F}_t]$ by definition of the Snell envelope U. Hence using the same observation as in (c), we have

$$\begin{aligned} U_{(t+1)\wedge\tau^*} - U_{t\wedge\tau^*} &= \mathbb{1}_{t+1\leq\tau^*}(U_{t+1} - U_t) \\ &= \mathbb{1}_{t+1\leq\tau^*}(U_{t+1} - E[U_{t+1}|\mathcal{F}_t]) \end{aligned}$$

Taking the expectations on both sides gives the martingale property of the process $(U_{t\wedge\tau^*})_{0\leq t\leq T}$. Integrability and adaptedness are trivial and hence the process U stopped at τ^* is thus a martingale. In particular, we have

$$E[U_{T\wedge\tau^*}] = U_0$$

by the OPtimional stopping Theorem. Moreover, since we assumed $U_T = Y_T$, one must have $\tau^* \leq T$ and hence

$$E[Y_{\tau^*}] = E[Y_{T\wedge\tau^*}] = E[U_{\wedge\tau^*}] = E[U_{\tau^*}] = U_0$$

(e) Since U is a supermartingale,

$$U_0 \geq E[U_\tau]$$

for any stopping time τ by the Optional Stopping Theorem. But since $U_t \geq Y_t$ by construction of the Snell envelope, we also have

$$U_0 \geq E[U_\tau] \geq E[Y_\tau]$$

Taking the supremum over stopping times $0 \leq \tau \leq T$ on both sides gives

$$U_0 \geq \sup\{E[Y_\tau] : \text{stopping times } 0 \leq \tau \leq T\}$$

For the other inequality we note that for $\tau^* = \min\{t \in \{0, ..., T\} \text{ such that } U_t = Y_t\}$, we have by the previous question

$$U_0 = E[Y_{\tau^*}]$$

and hence

$$U_0 \leq \sup\{E[Y_\tau] : \text{stopping times } 0 \leq \tau \leq T\}$$

(f) Follow directly from (d) and (e)

$$U_0 = E[Y_{\tau^*}] = \sup\{E[Y_\tau] : \text{stopping times } 0 \leq \tau \leq T\}$$

(g) Optimal exercise of American Options.

Exercise 5.3 *This is an optional exercise. You are highly encouraged to solved it, but the results of this exercise are not part of the exam material.* This exercise guides you through an alternative proof of the "hard" direction of the First Fundamental Theorem of Asset Pricing (also known as Dalang-Morton-Willinger Theorem). In this exercise we will focus on the basic one-period model, i.e we suppose that $T = 1$. The proof for the multi-period case is very similar but is a little more difficult because of some technicalities involving measurability. For simplicity, we also assume that $\mathcal{F}_0$ is (P-) trivial, so θ predictable means $\theta \in \mathbb{R}^d$. Moreover we also suppose that there exists a numéraire asset.

Let $(\tilde{S}_0^0, \tilde{S}_0^1)$ (respectively $(\tilde{S}_1^0, \tilde{S}_1^1)$) denote the vector of initial undiscounted prices (respectively terminal undiscounted prices), and let $(1, S_t^1)_{t\in\{0,1\}}$ be the discounted (with respect to the numéraire asset $\tilde{S}^0$) price process.

(a) Define a *pricing kernel* (also called *stochastic discount factor* or *state price density*) as a strictly positive random variable ρ satisfying

$$\tilde{S}_0^1 = \mathbb{E}_P\left[\rho \tilde{S}_1^1\right]$$

where P is the objective (or historical or statistical) measure of our filtered probability space $(\Omega, \mathcal{F}, P)$. When the market has a numéraire, we can characterize pricing kernels in terms of the discounted prices $(1, S^1)$: the pricing kernel ρ is a positive random variable $\rho > 0$ in $L^\infty(P)$ satisfying

$$E_P\left[\rho \Delta S_1^1\right] = 0$$

Show that when the market has a numéraire, the notion of a pricing kernel and that of an EMM are essentially the same. More precisely, show that the measure Q defined by

$$\frac{dQ}{dP} = \frac{\rho}{E_P[\rho]}$$

gives an EMM.

Since we suppose the existence of a numéraire, by a general result, the market is arbitrage free iff there is no arbitrage of the first kind, i.e. no arbitrage as defined in this lecture[1]. Moreover by question (a) the existence of a pricing kernel is equivalent to the existence of an EMM. We thus have to show that no arbitrage (of first kind) implies the existence of a pricing kernel ρ.

(b) Consider the function $F\colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ defined by

$$F(\theta) = \mathbb{E}_P\left[e^{-\theta\cdot\Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2}\right]$$

Show that F is finite valued and smooth (C^1).

(c) Suppose that there exists a minimiser θ^* of F. Construct a pricing kernel ρ and show that the corresponding EMM Q has a bounded Radon Nykodym derivative, i.e. $\frac{dQ}{dP} \in L^\infty$.

(d) In this question we show that the no arbitrage (of first kind) assumption implies the existence of a minimiser θ^* of F.

- Let $(\theta_k)_k$ be a minimizing sequence, i.e a sequence that satisfies

$$\lim_{k\to\infty} F(\theta_k) = \inf_{\theta\in\mathbb{R}^d} F(\theta)$$

  Suppose that $(\theta_k)_k$ is bounded. Show that in this case F admits a minimiser θ^*.

It remains to show that no arbitrage (of first kind) implies the existence of a bounded minimising sequence $(\theta_k)_k$.

Let $\mathcal{U} = \{\theta \in \mathbb{R}^d : \theta \cdot \Delta S_1^1 = 0 \ P\text{-}a.s\} \subseteq \mathbb{R}^d$ and $\mathcal{V} = \mathcal{U}^\perp$ the orthogonal complement of $\mathcal{U}$.

- Show that if $u \in \mathcal{U}$ and $v \in \mathcal{V}$ then $F(u + v) = F(v)$

Choose a minimising sequence $(\theta_k)_k$. By the previous result we can assume without loss of generality that $\theta_k \in \mathcal{V}$ for all k (otherwise we can construct a minimising sequence valued in $\mathcal{V}$ by projecting the original sequence $(\theta_k)_k$ on $\mathcal{V}$. The obtained projected sequence is still a minimising sequence since the projection does not change the value of the function $F(\cdot)$ by the previous question). Assume by contradiction that $(\theta_k)_k$ is unbounded, i.e after passing to a subsequence (again we continue to denote it by $(\theta_k)_k$), $||\theta_k|| \to \infty$. The goal of the next questions is to use the No Arbitrage assumption to get a contradiction.

[1] There are several types of arbitrages, but in this course we only study arbitrages of the first kind because we always assume the existence of a numéraire

- Since $(\theta_k)_k$ is unbounded, we can pass to a subsequence such that $||\theta_k|| \to \infty$. Define $\hat{\theta}_k = \frac{\theta_k}{||\theta_k||}$.
  Show that $\hat{\theta}_k \in \mathcal{V}$ and $||\hat{\theta}_k|| = 1$.

By Bolzano-Weierstrass Theorem, the bounded sequence $(\hat{\theta}_k)_k$ admits a converging subsequence. Let $\hat{\theta}_k$ denote this converging subsequence and let $\hat{\theta}$ be the limit of $\hat{\theta}_k$.

- Show that $\mathcal{V}$ is a closed set and conclude that $\hat{\theta} \in \mathcal{V}$. Show also that $\hat{\theta} \in \mathcal{V}$ and has unit norm.
- Show that the sequence $(F(\theta_k))_k$ is bounded.
- By showing that
$$F(\theta_k) = E_P\left[\left(e^{-\hat{\theta}_k \cdot \Delta S_1^1}\right)^{||\theta_k||} e^{-\frac{||\Delta S_1^1||^2}{2}}\right]$$
  conclude that we must have $\hat{\theta} \cdot \Delta S_1^1 \geq 0$ a.s.
- By using the no arbitrage assumption find a contradiction. Conclude that $(\theta_k)_k$ is bounded.

Solution 5.3

(a) This question simply tests your understanding of the definitions. Note that by Bayes formula we have
$$\begin{aligned}\mathbb{E}_Q\left[S_1^1\right] &= \mathbb{E}_P\left[\frac{dQ}{dP}S_1^1\right] \\ &= \mathbb{E}_P\left[\frac{\rho}{E_P[\rho]}S_1^1\right] \\ &= \frac{\mathbb{E}_P\left[\rho S_1^1\right]}{E_P[\rho]}\end{aligned}$$

If ρ is a pricing kernel then we get $\mathbb{E}_P\left[\rho S_1^1\right] = E_P[^1_0] = S_0^1 E_P[\rho]$ (where we used that $\mathcal{F}_0$ is trivial in the last equality) and so
$$\mathbb{E}_Q\left[S_1^1\right] = S_0^1$$

Moreover since $\rho > 0$ and $Q[\Omega] = E_Q[\mathbb{1}_\Omega] = E_P\left[\frac{\rho}{E_P[\rho]}\right] = 1$ and Q clearly statisfies the sigma additivity, we conclude that Q is indeed an EMM. *Note that one should rigorously prove the sigma additivity of Q. We leave it as an exercise since the same arguments can be used as in the previous two exercise sheets. In the exam, you are expected to write down the details*

On the other hand if Q is an EMM then $\mathbb{E}_Q\left[S_1^1\right] = S_0^1$ and so by the relation we derived above, the Radon Nykodym derivative $\frac{dQ}{dP}$ defines a pricing kernel up to normalization.

(b) $F(\cdot)$ is clearly finite valued since the integrand is bounded. Indeed
$$\begin{aligned}e^{-\theta\cdot\Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2} &\leq e^{-\theta\cdot\Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2} e^{-\frac{||\theta||^2}{2}} e^{\frac{||\theta||^2}{2}} \\ &= e^{-\frac{||\theta+\Delta S_1^1||^2}{2}} e^{\frac{||\theta||^2}{2}} \\ &\leq e^{\frac{||\theta||^2}{2}}\end{aligned}$$

where in the last equality we used that $||\theta + \Delta S_1^1||^2 \geq 0$ so $e^{-\frac{||\theta+\Delta S_1^1||^2}{2}} \leq 1$.

For the C^1 property of $F(\cdot)$, consider

$$f(\theta) := -\Delta S_1^1 \exp\left(-\theta \cdot \Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2\right)$$

We need to show that $f(\cdot)$ is locally bounded in θ. Indeed in that case we can exchange the gradient and expectation operators and we get $F(\cdot) \in C^1$ with

$$\nabla F(\theta) = E_P\left[-\Delta S_1^1 \exp\left(-\theta \cdot \Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2\right)\right]$$

We have by the Cauchy Schwarz inequality

$$\begin{aligned}|f(\theta)| &\leq ||\Delta S_1^1|| \exp\left(||\theta||\,||\Delta S_1^1|| - \frac{1}{2}||\Delta S_1^1||^2\right)\\ &\leq \sup_{\lambda\geq 0}\left[\lambda \exp\left(\lambda||\theta|| - \frac{\lambda^2}{2}\right)\right]\end{aligned}$$

The latter can be bounded as a function of θ hence $f(\cdot)$ is locally bounded in θ and thus $F(\cdot) \in C^1$.

(c) We have seen in question (b), that we can exchange the gradient and expectation operations to get

$$\nabla F(\theta) = E_P\left[-\Delta S_1^1 \exp\left(-\theta \cdot \Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2\right)\right]$$

Let θ^* be a minimiser of F. By the first order condition for a minimum, we have

$$\nabla F(\theta^*) = 0 = E_P\left[-\Delta S_1^1 \exp\left(-\theta^* \cdot \Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2\right)\right]$$

and hence $\rho = \exp\left(-\theta^* \cdot \Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2\right)$ is a pricing kernel.

Note that using the bound $e^{-\theta\cdot\Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2} \leq e^{\frac{||\theta||^2}{2}}$ obtained in b), we have

$$\begin{aligned}\rho &= \exp\left(-\theta^* \cdot \Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2\right)\\ &\leq \exp\left(\frac{||\theta^*||^2}{2}\right)\end{aligned}$$

and hence the Radon Nykodym derivative of the corresponding EMM Q, $\frac{dQ}{dP} = \frac{\rho}{E_P[\rho]}$, is in L^∞.

(d)
- If $(\theta_k)_k$ is bounded, then Bolzano Weierstrass Theorem gives us the existence of a converging subsequence. For notational simplicity we will continue to denote this sequence by $(\theta_k)_k$ and write θ^* for the limit of this converging subsequence. Since F is continuous,

  $$\lim_{k\to\infty} F(\theta_k) = F(\theta^*) = \inf_{\theta\in\mathbb{R}^d} F(\theta)$$

  where the last equality comes from the fact that $(\theta_k)_k$ is a minimizing sequence. Hence θ^* is a minimiser of F.

- Direct from the definition of F and $\mathcal{U}$:

$$\begin{aligned} F(u+v) &= \mathbb{E}_P\left[e^{-(u+v)\cdot\Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2}\right] \\ &= \mathbb{E}_P\left[e^{-v\cdot\Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2}\right] \\ &= F(v) \end{aligned}$$

  where the second equality uses that $u \in \mathcal{U}$.

- $||\hat{\theta}_k|| = 1$ by definition. Morever $\hat{\theta}_k \in \mathcal{V}$ since $\theta_k \in \mathcal{V}$. Indeed $\theta_k \in \mathcal{V}$ means that the scalar product between θ_k and any point in $\mathcal{U}$ is 0:

$$\theta_k \cdot \theta = 0 \quad \forall \theta \in \mathcal{U}$$

  This however directly implies

$$\hat{\theta}_k \cdot \theta = \frac{\theta_k \cdot \theta}{||\theta_k||} = 0 \quad \forall \theta \in \mathcal{U}$$

  and hence $\hat{\theta}_k \in \mathcal{V}$.

- By a standard result on finite-dimensional linear subspaces, $\mathcal{V}$ is closed. Hence $\hat{\theta} \in \mathcal{V}$ as the limit of the sequence $\hat{\theta}_k \in \mathcal{V}$. Another standard result tells that the inner product is a continuous map which implies that $||\hat{\theta}|| = ||\lim \hat{\theta}_k|| = \lim ||\hat{\theta}_k|| = 1$.

  Let's prove the above mentioned standard results. Let X be an inner product space (in our case $X = \mathbb{R}^d$). First we show that the inner product is a continuous map. Let $S_1^1, x_2, y_1, y_2 \in X$, by linearity of the inner product and Cauchy-Schwarz inequality we get,

$$\begin{aligned} |S_1^1 \cdot y_1 - x_2 \cdot y_2| &= |(S_1^1 - x_2)\cdot y_1 + x_2 \cdot (y_1 - y_2)| \\ &\leq ||S_1^1 - x_2||\ ||y_1|| + ||x_2||\ ||y_1 - y_2|| \end{aligned}$$

  This implies the continuity of inner products.

  Now let $A \subset X$. To show that $A^\perp$ is closed, consider a converging sequence (y_n) of elements of $A^\perp$ that converges to $y \in X$. We have to show that $y \in A^\perp$. Since the sequence (y_n) takes values in $A^\perp$, we have for all n

$$y_n \cdot a = 0 \ \forall a \in A$$

  hence

$$\lim_{n\to\infty}(y_n \cdot a) = 0 \ \forall a \in A$$

  But by continuity of the inner product,

$$\lim_{n\to\infty}(y_n \cdot a) = (\lim_{n\to\infty} y_n)\cdot a = y \cdot a = 0$$

  which shows that $A^\perp$ is closed.

- Since θ_k is a minimising sequence, there exists an index k_0 such that

$$|F(\theta_k)| = F(\theta_k) \leq F(0) + 1 \quad \forall k \geq k_0$$

  Indeed since θ_k is a minimising sequence,

$$\lim_{k\to\infty} F(\theta_k) = \inf_{\theta\in\mathbb{R}^d} F(\theta) \leq F(0)$$

and hence we must indeed have $|F(\theta_k)| = F(\theta_k) \leq F(0) + 1$ for k large enough (by the definition of inf). Finally

$$|F(\theta_k)| = F(\theta_k) \leq (F(0) + 1) \vee \max_{k \leq k_0} F(\theta_k) \quad \forall k$$

Hence the sequence $(F(\theta_k))_k$ is bounded.

- By definition

$$F(\theta) = \mathbb{E}_P \left[e^{-\theta \cdot \Delta S_1^1 - \frac{1}{2}||\Delta S_1^1||^2} \right]$$

Using that $\theta_k = \hat{\theta}_k ||\theta_k||$ we directly get

$$F(\theta_k) = E_P \left[\left(e^{-\hat{\theta}_k \cdot \Delta S_1^1} \right)^{||\theta_k||} e^{-\frac{||\Delta S_1^1||^2}{2}} \right]$$

Since $F(\theta_k)$ is bounded, we must have $\hat{\theta}_k \cdot \Delta S_1^1 \geq 0$ a.s (as otherwise the right-hand side of the above expression would blow up). By taking the limit, $\hat{\theta} \cdot \Delta S_1^1 \geq 0$ a.s.

- No arbitrage implies $\hat{\theta} \cdot \Delta S_1^1 = 0$ which means that $\hat{\theta} \in \mathcal{U}$. But we already saw that $\hat{\theta} \in \mathcal{V}$ and hence $\hat{\theta} \in \mathcal{U} \cap \mathcal{V} = \{0\}$. Note that the last equality comes from the fact that $\mathcal{V}$ is the orthogonal complement of $\mathcal{U}$. So in particular we must have $\hat{\theta} = 0$. But this contradicts the fact that $||\hat{\theta}|| = 1$.