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Mathematical Foundations for Finance

Exercise sheet 6

Please upload your solutions until Wednesday, 03/11/2021, 12:00 using the link on the course website.

Exercise 6.1 Consider a financial market in finite discrete time on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with undiscounted prices $\widetilde{S}^0$, $\widetilde{S}$ and discounted prices $1, S = \widetilde{S}/\widetilde{S}^0$. An arbitrage opportunity in the undiscounted market is a self-financing strategy φ with $\widetilde{V}(\varphi) \geq -a$ P-a.s. for some $a \geq 0$ (admissibility), $\widetilde{V}_0(\varphi) = 0$, $\widetilde{V}_T(\varphi) \geq 0$ P-a.s. and $P\left[\widetilde{V}_T(\varphi) > 0\right] > 0$.

(a) Show that $(\widetilde{S}^0, \widetilde{S})$ is free of arbitrage if and only if (S^0, S) is.
Hint: Remember that in finite discrete time we have that $NA \iff NA'$.

(b) Construct an example where $\widetilde{S}$ admits an EMM, but is not arbitrage-free. Does S then admit an EMM? What can you say about $\widetilde{S}^0$ for any such example?

(c) In your example, construct explicitly an arbitrage opportunity for the undiscounted market.

(d) Try to provide some intuition behind the existence of an EMM for $\widetilde{S}$ not implying (NA) when we know that the existence of an EMM for S does.

Exercise 6.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, P, \widetilde{S}^0, \widetilde{S}^1)$ be our canonical setup for a one-period trinomial model in which the evolution of $(\widetilde{S}^0, \widetilde{S}^1)$ is given by

$$\widetilde{S}^1_0 = S^1_0 = 80, \quad \widetilde{S}^1_1 = \begin{cases} 120 & \text{with probability } p_1 = 0.2, \\ 90 & \text{with probability } p_2 = 0.3, \\ 60 & \text{with probability } p_3 = 0.5 \end{cases}$$

$$\widetilde{S}^0_0 = 1, \quad \widetilde{S}^0_1 = 1 + 0.05.$$

(a) Check if the market is arbitrage-free by finding at least one EMM for $S^1 = \widetilde{S}^1/\widetilde{S}^0$.

(b) Find the set of all EMMs for S^1.

(c) Compute $E_Q\left[\frac{\widetilde{C}}{1+0.05}\right]$ for all $Q \in P_e(S^1)$, where $\widetilde{C}$ is the (undiscounted) payoff of a European call option with maturity $T = 1$ and strike price $\widetilde{K} = 80$, i.e. $\widetilde{C}(\omega) = (\widetilde{S}^1_1(\omega) - 80)^+$.

(d) Determine whether $\widetilde{C}$ as given in (c) is attainable.

(e) Find the set of all attainable payoffs $\widetilde{H} \in L^0_+(\mathcal{F}_1)$.
Hint: Every payoff is characterized by the values it takes on the atoms of $\mathcal{F}_1$. The set of all attainable payoffs can be identified with the set of solutions to a linear system.

Exercise 6.3 Consider the discounted market $(\Omega, \mathcal{F}, \mathbb{F}, P, 1, S^1)$ and assume that the stock price process is adapted to $\mathbb{F}$. Following points (a)–(c), show that $P_e(S^1)$, the set of all EMMs for S^1, is convex, i.e. that for all $Q_1, Q_2 \in P_e(S^1)$, the map $Q^\lambda : \mathcal{F} \to \mathbb{R}$ given by

$$Q^\lambda[A] = \lambda Q_1[A] + (1-\lambda) Q_2[A] \quad \text{for } A \in \mathcal{F}$$

is an EMM for S^1 for all $\lambda \in [0, 1]$.

(a) Show that Q^λ is a probability measure and that it is equivalent to P for all $\lambda \in [0, 1]$.

(b) Fix a $\lambda \in [0, 1]$. By the Radon–Nikodým theorem (see page 40 in the lecture notes), since Q^λ is a probability measures equivalent to P, there exists a density $\mathcal{D}^\lambda := \frac{dQ^\lambda}{dP} > 0$ P-a.s. such that

$$Q^\lambda[A] = E\big[\mathcal{D}^\lambda \mathbb{1}_A\big] \quad \forall\, A \in \mathcal{F}.$$

Write $\mathcal{D}^\lambda$ as a function of $\mathcal{D}^i := \frac{dQ_i}{dP}$ for $i = 1, 2$, the densities of Q_1 and Q_2 w.r.t. P, respectively, and deduce the form of the density process of Q^λ with respect to P.

(c) Conclude that Q^λ is an equivalent martingale measure for S^1 for each $\lambda \in [0, 1]$.