Mathematical Foundations for Finance

Exercise sheet 6

Please upload your solutions until Wednesday, 03/11/2021, 12:00 using the link on the course website.

Exercise 6.1 Consider a financial market in finite discrete time on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with undiscounted prices $\widetilde{S}^0$, $\widetilde{S}$ and discounted prices $1, S = \widetilde{S}/\widetilde{S}^0$. An arbitrage opportunity in the undiscounted market is a self-financing strategy φ with $\widetilde{V}(\varphi) \geq -a$ P-a.s. for some $a \geq 0$ (admissibility), $\widetilde{V}_0(\varphi) = 0$, $\widetilde{V}_T(\varphi) \geq 0$ P-a.s. and $P\left[\widetilde{V}_T(\varphi) > 0\right] > 0$.

(a) Show that $(\widetilde{S}^0, \widetilde{S})$ is free of arbitrage if and only if (S^0, S) is.
Hint: Remember that in finite discrete time we have that $NA \iff NA'$.

(b) Construct an example where $\widetilde{S}$ admits an EMM, but is not arbitrage-free. Does S then admit an EMM? What can you say about $\widetilde{S}^0$ for any such example?

(c) In your example, construct explicitly an arbitrage opportunity for the undiscounted market.

(d) Try to provide some intuition behind the existence of an EMM for $\widetilde{S}$ not implying (NA) when we know that the existence of an EMM for S does.

Solution 6.1

(a) As we have seen in Exercise 2.1, we have

$$\begin{aligned}\Delta\widetilde{C}_k(\varphi) &= \widetilde{C}_k(\varphi) - \widetilde{C}_{k-1}(\varphi)\\ &= (\varphi^0_k - \varphi^0_{k-1})\widetilde{S}^0_{k-1} + (\vartheta_k - \vartheta_{k-1})^{tr}\widetilde{S}_{k-1}\\ &= \widetilde{S}^0_{k-1}\big((\varphi^0_k - \varphi^0_{k-1}) + (\vartheta_k - \vartheta_{k-1})^{tr}S_{k-1}\big)\\ &= \widetilde{S}^0_{k-1}\Delta C_k(\varphi)\end{aligned}$$

and φ is self-financing if and only if $C(\varphi)$ or equivalently $\widetilde{C}(\varphi)$ is constant over time. So φ is self-financing for the discounted market if and only if it is self-financing for the undiscounted market. Because

$$\widetilde{V}(\varphi) = \widetilde{S}^0 V(\varphi) \text{ with } \widetilde{S}^0 > 0, \tag{1}$$

$\widetilde{V}_0(\varphi) = 0$, $\widetilde{V}_T(\varphi) \geq 0$ P-a.s. and $P\left[\widetilde{V}_T(\varphi) > 0\right] > 0$ if and only if the discounted counterpart $V(\varphi)$ satisfies these properties. Note that it is clear from (1) that if φ is admissible in the undiscounted market but $\widetilde{S}^0$ is unbounded, then $V(\varphi)$ is not bounded from below and φ therefore not admissible in the discounted market. So we have no equivalence between admissibility in the discounted and undiscounted market in general. However, we know from Exercise 5.3 (or equivalently the equivalence between NA and NA') that every strategy that satisfies all the defining properties of an arbitrage but the admissibility can be be modified into an admissible strategy while retaining all the other properties, which concludes the proof.

(b) Take $\widetilde{S} \equiv 1$ and $\widetilde{S}^0_k = (1+r)^k$ with $r > 0$. Then $\widetilde{S}$ is trivially a P-martingale, so P itself is an EMM for $\widetilde{S}$. However, $S_k = \widetilde{S}_k/\widetilde{S}^0_k = (1+r)^{-k}$ is strictly decreasing and hence does not admit an EMM. We know this about S because it is deterministic and since we require

$Q \approx P$, the attainable values of the process cannot change. So the market (S^0, S) is not arbitrage-free, and then neither is $(\widetilde{S}^0, \widetilde{S})$ by (a).

In particular, S does not admit an EMM and because $\widetilde{S}$ does, we can conclude that $\widetilde{S}^0$ cannot be identically equal to 1, since S and $\widetilde{S}$ then coincide.

(c) The idea is simple – short $\widetilde{S}$, long $\widetilde{S}^0$, i.e. $\varphi \equiv (1, -1)$ is obviously self-financing, $\widetilde{V}_0(\varphi) = 0$ and $\widetilde{V}_k(\varphi) = (1+r)^k - 1 > 0$.

(d) Since an arbitrage opportunity is a self-financing trading strategy by definition, it involves trading in at least two assets (we have to fund our position in one asset by the proceedings from the sale of another asset). The notion of arbitrage is thus necessarily tied to how assets behave in relation to each other. When we discount one process with another, we relate them to each other and look for a particular behavior (being a martingale under Q) in the process corresponding to relative performance. On the other hand, we cannot expect to find out some quality of the joint behavior of the processes by looking only at their individual behavior.

Exercise 6.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, P, \widetilde{S}^0, \widetilde{S}^1)$ be our canonical setup for a one-period trinomial model in which the evolution of $(\widetilde{S}^0, \widetilde{S}^1)$ is given by

$$\widetilde{S}^1_0 = S^1_0 = 80, \quad \widetilde{S}^1_1 = \begin{cases} 120 & \text{with probability } p_1 = 0.2, \\ 90 & \text{with probability } p_2 = 0.3, \\ 60 & \text{with probability } p_3 = 0.5 \end{cases}$$
$$\widetilde{S}^0_0 = 1, \quad \widetilde{S}^0_1 = 1 + 0.05.$$

(a) Check if the market is arbitrage-free by finding at least one EMM for $S^1 = \widetilde{S}^1/\widetilde{S}^0$.

(b) Find the set of all EMMs for S^1.

(c) Compute $E_Q\left[\frac{\widetilde{C}}{1+0.05}\right]$ for all $Q \in P_e(S^1)$, where $\widetilde{C}$ is the (undiscounted) payoff of a European call option with maturity $T = 1$ and strike price $\widetilde{K} = 80$, i.e. $\widetilde{C}(\omega) = (\widetilde{S}^1_1(\omega) - 80)^+$.

(d) Determine whether $\widetilde{C}$ as given in (c) is attainable.

(e) Find the set of all attainable payoffs $\widetilde{H} \in L^0_+(\mathcal{F}_1)$.
Hint: Every payoff is characterized by the values it takes on the atoms of $\mathcal{F}_1$. The set of all attainable payoffs can be identified with the set of solutions to a linear system.

Solution 6.2 We use the following notation:

$$S^1_0 = 80, \qquad \widetilde{S}^1_1 = \begin{cases} 80(1+y_1) & \text{with probability } p_1 = 0.2, \\ 80(1+y_2) & \text{with probability } p_2 = 0.3, \\ 80(1+y_3) & \text{with probability } p_3 = 0.5, \end{cases}$$

where $y_1 = \frac{1}{2}, y_2 = \frac{1}{8}, y_3 = -\frac{1}{4}$. Also let $q_i = Q\left[\left\{\widetilde{S}^1_1 = S^1_0(1+y_i)\right\}\right]$ and $r = 0.05$.

(a) We are working on a (finite) path space Ω and using the filtration generated by $\widetilde{S}^1$, which is also the filtration generated by S^1. S^1 is therefore clearly adapted and integrable. It is a martingale under a probability measure Q if and only if $E_Q\left[S_1^1\right] = S_0^1$.

$$E_Q\left[S_1^1\right] = S_0^1 \iff E_Q\left[\widetilde{S}_1^1\right] = S_0^1(1+r),$$

$$Q \text{ is an EMM} \iff \begin{cases} (1+y_1)q_1 + (1+y_2)q_2 + (1+y_3)q_3 = 1 + 0.05, \\ q_1 + q_2 + q_3 = 1, \\ (q_1, q_2, q_3) \in (0,1)^3. \end{cases}$$

$$\iff \begin{cases} \frac{1}{2}q_1 + \frac{1}{8}q_2 - \frac{1}{4}q_3 = 0.05, \\ q_1 + q_2 + q_3 = 1, \\ (q_1, q_2, q_3) \in (0,1)^3. \end{cases}$$

A particular solution to this system is for instance the vector $q_p = (0.2, 0.4, 0.4)$, which gives us by the fundamental theorem of asset pricing that the market S^1 is free of arbitrage.

(b) Picking up where we left off in (a), we can also easily compute that

$$Q \text{ is an EMM} \iff \begin{cases} q_2 = 0.8 - 2q_1 \\ q_3 = 0.2 + q_1 \\ q_1 \in (0, 0.4) \end{cases}$$

The set of all equivalent martingale measures can therefore by parametrized as

$$P_e(S^1) = \{Q_\alpha = (\alpha, 0.8 - 2\alpha, 0.2 + \alpha) \mid \alpha \in (0, 0.4)\}.$$

(c) Let us denote by $\mathcal{C}$ the set of all values that the expectation of $\frac{\widetilde{C}}{1+r}$ can attain under some $Q \in P_e(S^1)$. Then we compute

$$\begin{aligned} \mathcal{C} &= \left\{ E_{\mathbb{Q}_\alpha}\left[\frac{(\widetilde{S}_1^1 - 80)^+}{1+r}\right] \,\middle|\, \alpha \in (0, 0.4) \right\}, \\ &= \left\{ \frac{1}{1.05}\Big(40 \cdot \alpha + 10(0.8 - 2\alpha)\Big) \,\middle|\, \alpha \in (0, 0.4) \right\}, \\ &= \left\{ \frac{1}{1.05}(20\alpha + 8) \,\middle|\, \alpha \in (0, 0.4) \right\}, \\ &= (7.619, 15.238). \end{aligned}$$

(d) Since the map $Q_\alpha \mapsto E_{Q_\alpha}\left[\frac{\widetilde{C}}{1+r}\right]$ is clearly not constant, Theorem III.1.2 in the lecture notes gives us that the payoff is not attainable.

(e) $\widetilde{H} \in L_+^0(\mathcal{F}_T)$ can be replicated if and only if there exists an admissible self-financing trading strategy $\varphi = (\varphi^0, \vartheta)$ such that

$$\varphi_1^0(1+r) + \vartheta_1 \widetilde{S}_1^1 = \widetilde{H}$$

holds in every state of the world (that is P-a.s.). Note that this is obtained by taking the standard condition for the discounted value process $V(\varphi)$ and multiplying both sides of the equation by $(1+r)$ since the computation is then slightly more convenient. Let $\widetilde{H}^{y_i}$ be the value of the payoff if $\widetilde{S}_1^1 = S_0^1(1+y_i)$. We are looking for non-trivial solutions of the system

$$\begin{bmatrix} 1.05 & 120 \\ 1.05 & 90 \\ 1.05 & 60 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1^0 \\ \vartheta_1 \end{bmatrix} = \begin{bmatrix} \widetilde{H}^{y_1} \\ \widetilde{H}^{y_2} \\ \widetilde{H}^{y_3} \end{bmatrix}.$$

This system admits non-trivial solutions if and only if

$$\det \begin{bmatrix} 1.05 & 120 & \widetilde{H}^{y_1} \\ 1.05 & 90 & \widetilde{H}^{y_2} \\ 1.05 & 60 & \widetilde{H}^{y_3} \end{bmatrix} = 0.$$

$$\iff \quad 1.05 \times (90\widetilde{H}^{y_3} + 120\widetilde{H}^{y_2} + 60\widetilde{H}^{y_1} - 90\widetilde{H}^{y_1} - 60\widetilde{H}^{y_2} - 120\widetilde{H}^{y_3}) = 0$$

$$\iff \quad -\widetilde{H}^{y_1} + 2\widetilde{H}^{y_2} - \widetilde{H}^{y_3} = 0.$$

Note that this equation can also be used to verify that the payoff $\widetilde{C}$ from (c) is not attainable.

Exercise 6.3 Consider the discounted market $(\Omega, \mathcal{F}, \mathbb{F}, P, 1, S^1)$ and assume that the stock price process is adapted to $\mathbb{F}$. Following points (a)–(c), show that $P_e(S^1)$, the set of all EMMs for S^1, is convex, i.e. that for all $Q_1, Q_2 \in P_e(S^1)$, the map $Q^\lambda : \mathcal{F} \to \mathbb{R}$ given by

$$Q^\lambda[A] = \lambda Q_1[A] + (1-\lambda)Q_2[A] \quad \text{for } A \in \mathcal{F}$$

is an EMM for S^1 for all $\lambda \in [0,1]$.

(a) Show that Q^λ is a probability measure and that it is equivalent to P for all $\lambda \in [0,1]$.

(b) Fix a $\lambda \in [0,1]$. By the Radon–Nikodým theorem (see page 40 in the lecture notes), since Q^λ is a probability measures equivalent to P, there exists a density $\mathcal{D}^\lambda := \frac{dQ^\lambda}{dP} > 0$ P-a.s. such that

$$Q^\lambda[A] = E\big[\mathcal{D}^\lambda \mathbb{1}_A\big] \quad \forall\ A \in \mathcal{F}.$$

Write $\mathcal{D}^\lambda$ as a function of $\mathcal{D}^i := \frac{dQ_i}{dP}$ for $i = 1, 2$, the densities of Q_1 and Q_2 w.r.t. P, respectively, and deduce the form of the density process of Q^λ with respect to P.

(c) Conclude that Q^λ is an equivalent martingale measure for S^1 for each $\lambda \in [0,1]$.

Solution 6.3

(a) Fix a $\lambda \in [0,1]$. Recall that in order to be a probability measure equivalent to P, a map $Q : \mathcal{F} \to \mathbb{R}$ has to satisfy

(1) $Q[A] \geq 0$ for all $A \in \mathcal{F}$.

(2) $Q[\Omega] = 1$.

(3) For every collection of sets $(A_i)_{i=1}^\infty \subseteq \mathcal{F}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, we have that $Q\big[\bigcup_{i=1}^\infty A_i\big] = \sum_{i=1}^\infty Q[A_i]$.

(4) For each $A \in \mathcal{F}$, $Q[A] = 0$ if and only if $P[A] = 0$.

Since $Q_1, Q_2 \in P_e(S^1)$, they are probability measures equivalent to P, and thus satisfy (1)–(4). We show now that Q^λ satisfies (1)–(4), too. For the first property, note that for $A \in \mathcal{F}$,

$$Q^\lambda[A] = \lambda Q_1[A] + (1-\lambda)Q_2[A] \geq 0$$

since Q_i satisfies (1) and $\lambda \in [0,1]$ for $i = 1, 2$. Then, since Q_i satisfies (2) for $i = 1, 2$, we also have that $Q^\lambda[\Omega] = \lambda 1 + (1-\lambda)1 = 1$. For the third property, given a collection of sets $(A_i)_{i=1}^\infty \subseteq \mathcal{F}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, we can compute

$$Q^\lambda\left[\bigcup_{i=1}^\infty A_i\right] = \lambda \sum_{i=1}^\infty Q_1[A_i] + (1-\lambda)\sum_{i=1}^\infty Q_2[A_i] = \sum_{i=1}^\infty \big(\lambda Q_1[A_i] + (1-\lambda)Q_2[A_i]\big) = \sum_{i=1}^\infty Q^\lambda[A_i],$$

where for the first equality we use that Q_i satisfies (3). Finally, if $P[A] = 0$ for some $A \in \mathcal{F}$, we know by (4) that $Q_i[A] = 0$ for $i = 1, 2$, and hence $Q^\lambda[A] = \lambda 0 + (1-\lambda)0 = 0$. Analogously, if $P[A] > 0$ for some $A \in \mathcal{F}$, we know by (4) that $Q_i[A] > 0$ for $i = 1, 2$, and hence $Q^\lambda[A] = \lambda Q_1[A] + (1-\lambda)Q_2[A] > 0$, showing that Q^λ satisfies (4) and thus proving that it is a probability measure equivalent to P.

(b) By the definition of the density with respect to P, we have that

$$Q_1[A] = E[\mathcal{D}^1 \mathbb{1}_A] \qquad \text{and} \qquad Q_2[A] = E[\mathcal{D}^2 \mathbb{1}_A],$$

for all $A \in \mathcal{F}$. This implies that for all $A \in \mathcal{F}$

$$Q^\lambda[A] = \lambda E[\mathcal{D}^1 \mathbb{1}_A] + (1-\lambda)E[\mathcal{D}^2 \mathbb{1}_A] = E\big[(\lambda \mathcal{D}^1 + (1-\lambda)\mathcal{D}^2)\mathbb{1}_A\big],$$

and we can thus deduce that $\mathcal{D}^\lambda = \lambda \mathcal{D}^1 + (1-\lambda)\mathcal{D}^2$. Focusing on the density process $(Z_k^\lambda)_{k=0,1,\dots,T}$ of Q^λ with respect to P, we start by defining $Z^i = (Z_k^i)_{k=0,1,\dots,T}$ to be the density process of Q_i with respect to P, i.e.

$$Z_k^i := E[\mathcal{D}^i | \mathcal{F}_k]$$

for $i = 1, 2$. Then we can conclude that

$$Z_k^\lambda = E[\mathcal{D}^\lambda | \mathcal{F}_k] = \lambda E[\mathcal{D}^1 | \mathcal{F}_k] + (1-\lambda)E[\mathcal{D}^2 | \mathcal{F}_k] = \lambda Z_k^1 + (1-\lambda) Z_k^2.$$

(c) We already have that Q^λ is a probability measure equivalent to P, hence we only have to show that S^1 is a Q^λ-martingale. Since S^1 is a Q_i-martingale, we also know by Lemma II.3.1 that $S^1 Z^i$ is a P-martingale for $i = 1, 2$. Using that linear combinations of martingales are martingales, we thus directly obtain that the process

$$\lambda S^1 Z^1 + (1-\lambda) S^1 Z^2 = S^1 Z^\lambda$$

is a P-martingale, and hence that S^1 is a Q^λ-martingale.