

Mathematical Foundations for Finance

Exercise sheet 7

Please upload your solutions until Wednesday, 10/11/2021, 12:00 using the link on the course website.

Exercise 7.1 Let (Ω, \mathcal{F}, P) be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$, and let τ be an \mathbb{F} -stopping time. We define

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq k\} \in \mathcal{F}_k \text{ for all } k = 0, 1, \dots, T\}.$$

- (a) Show that \mathcal{F}_τ is a σ -algebra.
- (b) Suppose σ, τ are two stopping times with $\sigma \leq \tau$ P -a.s. Show that $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$. Show/conclude that if we set $\tau \equiv k_0$ for a fixed $k_0 \in \{0, 1, \dots, T\}$, then we have that $\mathcal{F}_\tau = \mathcal{F}_{k_0}$.
- (c) If τ, σ are two stopping times, then $\tau \vee \sigma$ and $\tau \wedge \sigma$ are stopping times, and $\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau \wedge \sigma}$. Moreover, $\{\sigma \leq \tau\} \in \mathcal{F}_{\tau \wedge \sigma}$ and $\{\sigma = \tau\} \in \mathcal{F}_{\tau \wedge \sigma}$.
- (d) Show that for a random variable $Y \in L_+^0(\mathcal{F})$, we have that

$$E[Y | \mathcal{F}_\tau] \mathbb{1}_{\{\tau=k\}} = E[Y | \mathcal{F}_k] \mathbb{1}_{\{\tau=k\}} \text{ } P\text{-a.s. for all } k \in \{0, 1, \dots, T\},$$

i.e. that $E[Y | \mathcal{F}_\tau] = E[Y | \mathcal{F}_k]$ P -a.s. on the set $\{\tau = k\}$ or, equivalently,

$$E[Y | \mathcal{F}_\tau] = \sum_{k=0}^T \mathbb{1}_{\{\tau=k\}} E[Y | \mathcal{F}_k] \text{ } P\text{-a.s.}$$

Exercise 7.2 Let $(\tilde{S}^0, \tilde{S}^1)$ be an *arbitrage-free* financial market with time horizon T and assume that the bank account process $\tilde{S}^0 = (\tilde{S}_k^0)_{k=0,1,\dots,T}$ is given by $\tilde{S}_k^0 = (1+r)^k$ for a constant $r \geq 0$. Denote the set of all EMMs for S^1 by $\mathbb{P}_e(S^1)$. Fix a $\tilde{K} > 0$. The undiscounted payoff of a *European call option* on \tilde{S}^1 with strike \tilde{K} and maturity $k \in \{1, \dots, T\}$ is denoted by \tilde{C}_k^E and given by

$$\tilde{C}_k^E = (\tilde{S}_k^1 - \tilde{K})^+,$$

whereas the undiscounted payoff of an *Asian call option* on \tilde{S}^1 with strike \tilde{K} and maturity $k \in \{1, \dots, T\}$ is denoted by \tilde{C}_k^A and given by

$$\tilde{C}_k^A := \left(\frac{1}{k} \sum_{j=1}^k \tilde{S}_j^1 - \tilde{K} \right)^+.$$

- (a) Fix a $Q \in \mathbb{P}_e(S^1)$ and show that the function $\{1, \dots, T\} \rightarrow \mathbb{R}_+, k \mapsto E_Q \left[\frac{\tilde{C}_k^E}{\tilde{S}_k^0} \right]$ is increasing.
Hint: Use Jensen's inequality for conditional expectations.
- (b) Fix a $Q \in \mathbb{P}_e(S^1)$ and show that for all $k = 1, \dots, T$, we have

$$E_Q \left[\frac{\tilde{C}_k^A}{\tilde{S}_k^0} \right] \leq \frac{1}{k} \sum_{j=1}^k E_Q \left[\frac{\tilde{C}_j^E}{\tilde{S}_j^0} \right].$$

(c) Fix a $Q \in \mathbb{P}_e(S^1)$ and deduce that for all $k = 1, \dots, T$, we have

$$E_Q \left[\frac{\tilde{C}_k^A}{\tilde{S}_k^0} \right] \leq E_Q \left[\frac{\tilde{C}_k^E}{\tilde{S}_k^0} \right].$$

Interpret this inequality.

Exercise 7.3 Let $(\tilde{S}^0, \tilde{S}^1)$ follow a binomial model with $\tilde{S}_0^1 = 1$, $u > r > d > -1$ and $T \in \mathbb{N}$. Denote by (\hat{S}^0, \hat{S}^1) the market discounted with \tilde{S}^1 , i.e.

$$\hat{S}^0 := \frac{\tilde{S}^0}{\tilde{S}^1} \quad \text{and} \quad \hat{S}^1 := \frac{\tilde{S}^1}{\tilde{S}^1} \equiv 1.$$

- (a) Show that there exists a unique equivalent martingale measure Q^{**} for \hat{S}^0 .
- (b) Let Q^* be the unique equivalent martingale measure for $S^1 = \tilde{S}^1/\tilde{S}^0$. Show that the density of Q^{**} with respect to Q^* on \mathcal{F}_T is given by

$$\frac{dQ^{**}}{dQ^*} = S_T^1.$$

(c) Show that for an *undiscounted* payoff $\tilde{H} \in L_+^0(\mathcal{F}_T)$, we have

$$\tilde{S}_k^0 E_{Q^*} \left[\frac{\tilde{H}}{\tilde{S}_T^0} \middle| \mathcal{F}_k \right] = \tilde{S}_k^1 E_{Q^{**}} \left[\frac{\tilde{H}}{\tilde{S}_T^1} \middle| \mathcal{F}_k \right], \quad k = 0, \dots, T.$$

This formula shows that the martingale pricing method is invariant under a so-called *change of numéraire*.

Hint: Use Bayes' formula (Lemma II.3.1) in the lecture notes.