

Mathematical Foundations for Finance

Exercise sheet 7

Please upload your solutions until Wednesday, 10/11/2021, 12:00 using the link on the course website.

Exercise 7.1 Let (Ω, \mathcal{F}, P) be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$, and let τ be an \mathbb{F} -stopping time. We define

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq k\} \in \mathcal{F}_k \text{ for all } k = 0, 1, \dots, T\}.$$

- (a) Show that \mathcal{F}_τ is a σ -algebra.
- (b) Suppose σ, τ are two stopping times with $\sigma \leq \tau$ P -a.s. Show that $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$. Show/conclude that if we set $\tau \equiv k_0$ for a fixed $k_0 \in \{0, 1, \dots, T\}$, then we have that $\mathcal{F}_\tau = \mathcal{F}_{k_0}$.
- (c) If τ, σ are two stopping times, then $\tau \vee \sigma$ and $\tau \wedge \sigma$ are stopping times, and $\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau \wedge \sigma}$. Moreover, $\{\sigma \leq \tau\} \in \mathcal{F}_{\tau \wedge \sigma}$ and $\{\sigma = \tau\} \in \mathcal{F}_{\tau \wedge \sigma}$.
- (d) Show that for a random variable $Y \in L^0_+(\mathcal{F})$, we have that

$$E[Y | \mathcal{F}_\tau] \mathbf{1}_{\{\tau=k\}} = E[Y | \mathcal{F}_k] \mathbf{1}_{\{\tau=k\}} \text{ } P\text{-a.s. for all } k \in \{0, 1, \dots, T\},$$

i.e. that $E[Y | \mathcal{F}_\tau] = E[Y | \mathcal{F}_k]$ P -a.s. on the set $\{\tau = k\}$ or, equivalently,

$$E[Y | \mathcal{F}_\tau] = \sum_{k=0}^T \mathbf{1}_{\{\tau=k\}} E[Y | \mathcal{F}_k] \text{ } P\text{-a.s.}$$

Solution 7.1

- (a) By the definition of a σ -algebra, we need to verify that $\Omega \in \mathcal{F}_\tau$, that if $A \in \mathcal{F}_\tau$, then also $A^c \in \mathcal{F}_\tau$, and that if $(A_n)_{n \in \mathbb{N}}$ is a sequence of sets in \mathcal{F}_τ , then also $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}_\tau$.

1. We have that $\Omega \cap \{\tau \leq k\} = \{\tau \leq k\} \in \mathcal{F}_k$ for all $k \in \{0, 1, \dots, T\}$, so $\Omega \in \mathcal{F}_\tau$.
2. For an $A \in \mathcal{F}_\tau$, we have

$$A^c \cap \{\tau \leq k\} = \{\tau \leq k\} \cap (A^c \cup \{\tau \leq k\}^c) = \{\tau \leq k\} \cap (A \cap \{\tau \leq k\})^c \in \mathcal{F}_k$$

for all $k \in \{0, 1, \dots, T\}$, so $A^c \in \mathcal{F}_\tau$.

3. We show that if $(A_n)_{n \in \mathbb{N}} \in \mathcal{F}_\tau$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}_\tau$. Indeed,

$$\left(\bigcup_{n=1}^{\infty} A_n \right) \cap \{\tau \leq k\} = \bigcup_{n=1}^{\infty} (A_n \cap \{\tau \leq k\}) \in \mathcal{F}_k$$

for all $k \in \{0, 1, \dots, T\}$, so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_\tau$. Hence \mathcal{F}_τ is a σ -algebra.

- (b) Let $A \in \mathcal{F}_\sigma$. The assumption $\sigma \leq \tau$ implies $\{\tau \leq k\} \subseteq \{\sigma \leq k\}$. Then for all $k \in \mathbb{N}_0$, we have

$$A \cap \{\tau \leq k\} = \underbrace{(A \cap \{\sigma \leq k\})}_{\in \mathcal{F}_k} \cap \{\tau \leq k\} \in \mathcal{F}_k$$

because $A \in \mathcal{F}_\sigma$. This shows $A \in \mathcal{F}_\tau$ and $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

Now if $\tau \equiv k_0$, then $\mathcal{F}_\tau \subseteq \mathcal{F}_{k_0}$ and $\mathcal{F}_{k_0} \subseteq \mathcal{F}_\tau$, which yields $\mathcal{F}_\tau = \mathcal{F}_{k_0}$. Alternatively one can show the last point using the same argument as above: for any $A \in \mathcal{F}$, we have that

$$A \cap \{\tau \leq k\} = A \cap \{k_0 \leq k\} = \begin{cases} A & \text{for } k \geq k_0, \\ \emptyset & \text{for } k < k_0. \end{cases}$$

If $A \in \mathcal{F}_\tau$, then all these sets are in \mathcal{F}_k , and we get $A \in \mathcal{F}_{k_0}$ for $k = k_0$. Conversely, if $A \in \mathcal{F}_{k_0}$, then all these sets are in \mathcal{F}_k for all $k \in \{0, 1, \dots, T\}$, so that $A \in \mathcal{F}_\tau$.

- (c) The claim that $\sigma \vee \tau, \sigma \wedge \tau$ are stopping times follow from the relations

$$\{\sigma \vee \tau \leq k\} = \{\sigma \leq k\} \cap \{\tau \leq k\} \in \mathcal{F}_k$$

and

$$\{\sigma \wedge \tau \leq k\} = \{\sigma \leq k\} \cup \{\tau \leq k\} \in \mathcal{F}_k.$$

Now because $\sigma \wedge \tau \leq \sigma$ and $\sigma \wedge \tau \leq \tau$, part (b) gives $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\sigma \cap \mathcal{F}_\tau$. Next suppose that $A \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$. We observe that

$$\begin{aligned} A \cap \{\sigma \wedge \tau \leq k\} &= A \cap (\{\sigma \leq k\} \cup \{\tau \leq k\}) \\ &= \underbrace{(A \cap \{\sigma \leq k\})}_{\in \mathcal{F}_k} \cup \underbrace{(A \cap \{\tau \leq k\})}_{\in \mathcal{F}_k} \in \mathcal{F}_k. \end{aligned}$$

This shows $\mathcal{F}_\sigma \cap \mathcal{F}_\tau \subseteq \mathcal{F}_{\sigma \wedge \tau}$ and hence $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

To prove the remaining claims, note that for each $k \in \mathbb{N}$,

$$\{\sigma \leq \tau\} \cap \{\tau \leq k\} = \bigcup_{i=0}^k (\{\sigma \leq \tau\} \cap \{\tau = i\}) = \bigcup_{i=0}^k (\{\sigma \leq i\} \cap \{\tau = i\}) \in \mathcal{F}_k.$$

Thus $\{\sigma \leq \tau\} \in \mathcal{F}_\tau$. Similarly, for each $k \in \mathbb{N}_0$, we have

$$\{\sigma \leq \tau\} \cap \{\sigma \leq k\} = \{\sigma \wedge k \leq \tau \wedge k\} \cap \{\sigma \leq k\} \in \mathcal{F}_k$$

because $\sigma \wedge k, \tau \wedge k$ are both \mathcal{F}_k -measurable by parts (a) and (b). Hence $\{\sigma \leq \tau\} \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma \wedge \tau}$.

The very last assertion follows from $\{\sigma = \tau\} = \{\sigma \leq \tau\} \cap \{\tau \leq \sigma\}$.

- (d) Call LHS and RHS the left- and right-hand sides of

$$E[Y | \mathcal{F}_\tau] \mathbf{1}_{\{\tau=k\}} = E[Y | \mathcal{F}_k] \mathbf{1}_{\{\tau=k\}},$$

respectively. Now RHS is \mathcal{F}_k -measurable since $\{\tau = k\} \in \mathcal{F}_k$. Moreover, $\{\tau = k\} \in \mathcal{F}_\tau$ since for all $\ell \in \{0, 1, \dots, T\}$,

$$\{\tau = k\} \cap \{\tau \leq \ell\} = \begin{cases} \{\tau = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_\ell & \text{for } k \leq \ell, \\ \emptyset \in \mathcal{F}_\ell & \text{for } k > \ell. \end{cases}$$

So the random variable $\mathbf{1}_{\{\tau=k\}}$ is \mathcal{F}_τ -measurable and

$$\text{LHS} = E[Y \mathbf{1}_{\{\tau=k\}} | \mathcal{F}_\tau].$$

Take any $A \in \mathcal{F}_\tau$ and note that $A \cap \{\tau = k\} \in \mathcal{F}_k$. Then,

$$\begin{aligned} E[Y \mathbf{1}_{\{\tau=k\}} \mathbf{1}_A] &= E[Y \mathbf{1}_{A \cap \{\tau=k\}}] = E[E[Y | \mathcal{F}_k] \mathbf{1}_{A \cap \{\tau=k\}}] \\ &= E[E[Y | \mathcal{F}_k] \mathbf{1}_{\{\tau=k\}} \mathbf{1}_A] = E[\text{RHS} \mathbf{1}_A], \end{aligned}$$

which shows that

$$\text{RHS} = E[Y \mathbf{1}_{\{\tau=k\}} | \mathcal{F}_\tau] = \text{LHS } P\text{-a.s.}$$

Notice that in this exercise we did not assume Y to be integrable (or even finite). The integral is nonetheless uniquely defined, but it is good to keep in mind that the (conditional) expectation is only an additive operator on $L_+^0(\mathcal{F})$ – it is not homogeneous, hence it is not linear.

Exercise 7.2 Let $(\tilde{S}^0, \tilde{S}^1)$ be an *arbitrage-free* financial market with time horizon T and assume that the bank account process $\tilde{S}^0 = (\tilde{S}_k^0)_{k=0,1,\dots,T}$ is given by $\tilde{S}_k^0 = (1+r)^k$ for a constant $r \geq 0$. Denote the set of all EMMs for S^1 by $\mathbb{P}_e(S^1)$. Fix a $\tilde{K} > 0$. The undiscounted payoff of a *European call option* on \tilde{S}^1 with strike \tilde{K} and maturity $k \in \{1, \dots, T\}$ is denoted by \tilde{C}_k^E and given by

$$\tilde{C}_k^E = (\tilde{S}_k^1 - \tilde{K})^+,$$

whereas the undiscounted payoff of an *Asian call option* on \tilde{S}^1 with strike \tilde{K} and maturity $k \in \{1, \dots, T\}$ is denoted by \tilde{C}_k^A and given by

$$\tilde{C}_k^A := \left(\frac{1}{k} \sum_{j=1}^k \tilde{S}_j^1 - \tilde{K} \right)^+.$$

(a) Fix a $Q \in \mathbb{P}_e(S^1)$ and show that the function $\{1, \dots, T\} \rightarrow \mathbb{R}_+$, $k \mapsto E_Q \left[\frac{\tilde{C}_k^E}{\tilde{S}_k^0} \right]$ is increasing.

Hint: Use Jensen's inequality for conditional expectations.

(b) Fix a $Q \in \mathbb{P}_e(S^1)$ and show that for all $k = 1, \dots, T$, we have

$$E_Q \left[\frac{\tilde{C}_k^A}{\tilde{S}_k^0} \right] \leq \frac{1}{k} \sum_{j=1}^k E_Q \left[\frac{\tilde{C}_j^E}{\tilde{S}_j^0} \right].$$

(c) Fix a $Q \in \mathbb{P}_e(S^1)$ and deduce that for all $k = 1, \dots, T$, we have

$$E_Q \left[\frac{\tilde{C}_k^A}{\tilde{S}_k^0} \right] \leq E_Q \left[\frac{\tilde{C}_k^E}{\tilde{S}_k^0} \right].$$

Interpret this inequality.

Solution 7.2

(a) It clearly suffices to show that for all $k = 1, \dots, T-1$, we have

$$E_Q \left[\frac{\tilde{C}_{k+1}^E}{\tilde{S}_{k+1}^0} \right] \geq E_Q \left[\frac{\tilde{C}_k^E}{\tilde{S}_k^0} \right].$$

Fix a $k \in \{1, \dots, T-1\}$. Using the *tower property* of conditional expectation, *Jensen's inequality* for conditional expectations (for the convex function $x \mapsto x^+$), the fact that S^1 is a

Q -martingale and that $r \geq 0$, we get

$$\begin{aligned}
E_Q \left[\frac{\tilde{C}_{k+1}^E}{\tilde{S}_{k+1}^0} \right] &= E_Q \left[\left(S_{k+1}^1 - \frac{\tilde{K}}{(1+r)^{k+1}} \right)^+ \right] \\
&= E_Q \left[E_Q \left[\left(S_{k+1}^1 - \frac{\tilde{K}}{(1+r)^{k+1}} \right)^+ \middle| \mathcal{F}_k \right] \right] \\
&\geq E_Q \left[\left(E_Q \left[S_{k+1}^1 - \frac{\tilde{K}}{(1+r)^{k+1}} \middle| \mathcal{F}_k \right] \right)^+ \right] \\
&= E_Q \left[\left(S_k^1 - \frac{\tilde{K}}{(1+r)^{k+1}} \right)^+ \right] \\
&\geq E_Q \left[\left(S_k^1 - \frac{\tilde{K}}{(1+r)^k} \right)^+ \right] \\
&= E_Q \left[\frac{\tilde{C}_k^E}{\tilde{S}_k^0} \right].
\end{aligned}$$

(b) Since the function $x \mapsto x^+$ is convex, we have for $k = 1, \dots, T$

$$\begin{aligned}
\tilde{C}_k^A &= \left(\frac{1}{k} \sum_{j=1}^k \tilde{S}_j^1 - \tilde{K} \right)^+ = \left(\sum_{j=1}^k \frac{1}{k} (\tilde{S}_j^1 - \tilde{K}) \right)^+ \\
&\leq \sum_{j=1}^k \frac{1}{k} (\tilde{S}_j^1 - \tilde{K})^+ = \frac{1}{k} \sum_{j=1}^k \tilde{C}_j^E.
\end{aligned}$$

By *linearity* and *monotonicity* of expectations and since $r \geq 0$, we get

$$\begin{aligned}
E_Q \left[\frac{\tilde{C}_k^A}{\tilde{S}_k^0} \right] &= E_Q \left[\frac{\tilde{C}_k^A}{(1+r)^k} \right] \leq \frac{1}{k} \sum_{j=1}^k E_Q \left[\frac{\tilde{C}_j^E}{(1+r)^k} \right] \\
&\leq \frac{1}{k} \sum_{j=1}^k E_Q \left[\frac{\tilde{C}_j^E}{(1+r)^j} \right] = \frac{1}{k} \sum_{j=1}^k E_Q \left[\frac{\tilde{C}_j^E}{\tilde{S}_j^0} \right].
\end{aligned}$$

(c) Putting the results of (a) and (b) together yields for $k = 1, \dots, T$

$$E_Q \left[\frac{\tilde{C}_k^A}{\tilde{S}_k^0} \right] \leq \frac{1}{k} \sum_{j=1}^k E_Q \left[\frac{\tilde{C}_j^E}{\tilde{S}_j^0} \right] \leq \frac{1}{k} \sum_{j=1}^k E_Q \left[\frac{\tilde{C}_k^E}{\tilde{S}_k^0} \right] = E_Q \left[\frac{\tilde{C}_k^E}{\tilde{S}_k^0} \right].$$

This means nothing else than that for a fixed EMM, the price of an Asian call option on \tilde{S}^1 is smaller than the price of the European call option on the same asset with the same strike price \tilde{K} and maturity $k \in \{1, \dots, T\}$.

This makes sense also intuitively since the price of a call option is increasing in the volatility of the underlying \tilde{A} (the probability of ending up in the money, i.e. $P[\tilde{A} > \tilde{K}]$, is higher), and averaging in the Asian call option amounts to reducing volatility of the underlying.

Exercise 7.3 Let $(\tilde{S}^0, \tilde{S}^1)$ follow a binomial model with $\tilde{S}_0^1 = 1$, $u > r > d > -1$ and $T \in \mathbb{N}$. Denote by (\hat{S}^0, \hat{S}^1) the market discounted with \tilde{S}^1 , i.e.

$$\hat{S}^0 := \frac{\tilde{S}^0}{\tilde{S}^1} \quad \text{and} \quad \hat{S}^1 := \frac{\tilde{S}^1}{\tilde{S}^1} \equiv 1.$$

- (a) Show that there exists a unique equivalent martingale measure Q^{**} for \hat{S}^0 .
- (b) Let Q^* be the unique equivalent martingale measure for $S^1 = \tilde{S}^1/\tilde{S}^0$. Show that the density of Q^{**} with respect to Q^* on \mathcal{F}_T is given by

$$\frac{dQ^{**}}{dQ^*} = S_T^1.$$

- (c) Show that for an *undiscounted* payoff $\tilde{H} \in L_+^0(\mathcal{F}_T)$, we have

$$\tilde{S}_k^0 E_{Q^*} \left[\frac{\tilde{H}}{\tilde{S}_T^0} \middle| \mathcal{F}_k \right] = \tilde{S}_k^1 E_{Q^{**}} \left[\frac{\tilde{H}}{\tilde{S}_T^1} \middle| \mathcal{F}_k \right], \quad k = 0, \dots, T.$$

This formula shows that the martingale pricing method is invariant under a so-called *change of numéraire*.

Hint: Use Bayes' formula (Lemma II.3.1) in the lecture notes.

Solution 7.3 Assume without loss of generality that $\Omega = \{1, 2\}^T$ and $Y_k((x_1, \dots, x_T)) = 1 + y_{x_k}$, where $y_1 := d$ and $y_2 := u$.

- (a) **First method.**

Any measure $Q \approx P$ on \mathcal{F}_T can be described by its transition probabilities $q_{x_1}, q_{x_1, x_2}, \dots, q_{x_1, \dots, x_T}$, where $x_1, \dots, x_k \in \{1, 2\}$ and

$$\begin{aligned} q_{x_1} &:= Q[Y_1 = 1 + y_{x_1}], \\ q_{x_1, \dots, x_k} &:= Q[Y_k = 1 + y_{x_k} \mid Y_1 = 1 + y_{x_1}, \dots, Y_{k-1} = 1 + y_{x_{k-1}}], \quad k = 2, \dots, T. \end{aligned}$$

Since \hat{S}^0 only attains a finite number of values and hence is bounded and since it is also adapted, it is a Q -martingale if and only if for all $k = 0, \dots, T - 1$, we have

$$E_Q \left[\hat{S}_{k+1}^0 \middle| \mathcal{F}_k \right] = \hat{S}_k^0 \quad Q\text{-a.s.} \tag{1}$$

By the definition of \hat{S}^0 and since it is strictly positive, the martingale property (1) holds if and only if

$$E_Q \left[\hat{S}_k^0 \frac{1+r}{Y_{k+1}} \middle| \mathcal{F}_k \right] = \hat{S}_k^0 \quad Q\text{-a.s.},$$

which holds if and only if

$$E_Q \left[\frac{1+r}{Y_{k+1}} \middle| \mathcal{F}_k \right] = 1 \quad Q\text{-a.s.}$$

Note that we do not know a priori whether the Y_k are independent under Q . Since $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_k = \sigma(Y_1, \dots, Y_k)$ for $k = 1, \dots, T$, and Y_k only takes two values, \hat{S}^0 is a Q -martingale if and only if $E_Q \left[\frac{1+r}{Y_1} \right] = 1$ and for all $k \in \{1, \dots, T - 1\}$ and $x_1, \dots, x_k \in \{1, 2\}$ we have

$$E_Q \left[\frac{1+r}{Y_{k+1}} \middle| Y_1 = 1 + y_{x_1}, \dots, Y_k = 1 + y_{x_k} \right] = 1.$$

Now, we have

$$\begin{aligned}
 E_Q \left[\frac{1+r}{Y_1} \right] = 1 &\iff \frac{1+r}{1+d} q_1 + \frac{1+r}{1+u} (1-q_1) = 1 \\
 &\iff \left(\frac{1+u}{1+d} - 1 \right) q_1 = \frac{1+u}{1+r} - 1 \\
 &\iff \frac{u-d}{1+d} q_1 = \frac{u-r}{1+r} \\
 &\iff q_1 = \frac{1+d}{1+r} \frac{u-r}{u-d}.
 \end{aligned}$$

Similarly, for all $k \in \{1, \dots, T-1\}$ and all $x_1, \dots, x_k \in \{1, 2\}$, we have

$$E_Q \left[\frac{1+r}{Y_{k+1}} \mid Y_1 = 1 + y_{x_1}, \dots, Y_k = 1 + y_{x_k} \right] = 1 \iff q_{x_1, \dots, x_k, 1} = \frac{1+d}{1+r} \frac{u-r}{u-d}.$$

Note that $q_{x_1, \dots, x_k, 1}$ does not depend on x_1, \dots, x_k and equals q_1 . Hence, we may conclude that there exists a unique equivalent martingale measure Q^{**} for \widehat{S}^0 , under which Y_1, \dots, Y_k are i.i.d. and we have

$$Q^{**}[Y = 1+d] = \frac{1+d}{1+r} \frac{u-r}{u-d} =: q_1^{**} \quad \text{and} \quad Q^{**}[Y = 1+u] = \frac{1+u}{1+r} \frac{r-d}{u-d} =: q_2^{**}.$$

Second method:

For $k = 1, \dots, T$, set $\widehat{Y}_k = \frac{1+r}{Y_k}$. Then $\widehat{S}_k^1 = 1^k = 1$ and $\widehat{S}_k^0 = \prod_{j=1}^k \widehat{Y}_j$ for $k = 0, \dots, T$, where the \widehat{Y}_k are independent under P and take the two values $1 + \hat{u}$ and $1 + \hat{d}$ with probability p_1 and p_2 , respectively, where $\hat{u} = \frac{r-d}{1+d} > 0$ and $\hat{d} = \frac{r-u}{1+u} < 0$. In conclusion, $(\widehat{S}^1, \widehat{S}^0)$ can be viewed as a binomial model with $\hat{u} > \hat{r} = 0 > \hat{d}$.

Variant (i):

Recalling that \hat{u} corresponds to p_1 and \hat{d} to p_2 , it follows from Corollary 2.1.4 in the lecture notes that the unique equivalent martingale measure Q^{**} for \widehat{S}^1 is given by

$$Q^{**}[\{(x_1, \dots, x_T)\}] := \prod_{k=1}^T q_{x_k}^{**}, \quad x_1, \dots, x_T \in \{1, 2\},$$

where

$$\begin{aligned}
 q_1^{**} &= \frac{\hat{r} - \hat{d}}{\hat{u} - \hat{d}} = \frac{\frac{u-r}{1+u}}{\frac{r-d}{1+d} - \frac{r-u}{1+u}} = \frac{1+d}{1+r} \frac{u-r}{u-d} \\
 q_2^{**} &= \frac{\hat{u} - \hat{r}}{\hat{u} - \hat{d}} = \frac{\frac{r-d}{1+d}}{\frac{r-d}{1+d} - \frac{r-u}{1+u}} = \frac{1+u}{1+r} \frac{r-d}{u-d}.
 \end{aligned}$$

Variant (ii):

Since we have a binomial model, we know there can be at most one EMM. So we are looking for the unique strictly positive Q^* -martingale $Z^{Q^{**}; Q^*}$ starting at 1 such that

$$\frac{\widetilde{S}^0}{\widetilde{S}^1} Z^{Q^{**}; Q^*} \text{ is a } Q^*\text{-martingale.}$$

But we already know that $\frac{\widetilde{S}^1}{\widetilde{S}^0}$ is a Q^* -martingale and strictly positive. Hence by uniqueness,

$$Z^{Q^{**}; Q^*} = \frac{\widetilde{S}^1}{\widetilde{S}^0} = S^1.$$

- (b) As explained in part (a), variant (ii), the density process $Z^{Q^{**};Q^*}$ of the unique EMM Q^{**} with respect to Q^* is given by

$$Z_k^{Q^{**};Q^*} = S_k^1, \quad \forall k \in \{0, \dots, T\}.$$

By Lemma II.3.1, this give us that for all $A \in \mathcal{F}_T$

$$Q^{**}[A] = E_{Q^*}[Z_T^{Q^{**};Q^*} \mathbf{1}_A] = E_{Q^*}[S_T^1 \mathbf{1}_A]$$

and thus that on \mathcal{F}_T ,

$$\frac{dQ^{**}}{dQ^*} = S_T^1.$$

- (c) In order to simplify the notation, denote by Z the density process $Z^{Q^{**};Q^*}$ of Q^{**} with respect to Q^* . Let now $k \in \{1, \dots, T\}$ and $\tilde{H} \in L_+^0(\mathcal{F}_T)$ and recall that

$$Z = S^1 = \frac{\tilde{S}^1}{\tilde{S}^0} \quad Q^*\text{-a.s.}$$

Thus, by the Bayes formula we get for $k = 0, \dots, T$

$$\tilde{S}_k^1 E_{Q^{**}} \left[\frac{\tilde{H}}{\tilde{S}_T^1} \middle| \mathcal{F}_k \right] = \frac{\tilde{S}_k^1}{Z_k} E_{Q^*} \left[\frac{Z_T \tilde{H}}{\tilde{S}_T^1} \middle| \mathcal{F}_k \right] = \tilde{S}_k^0 E_{Q^*} \left[\frac{\tilde{H}}{\tilde{S}_T^0} \middle| \mathcal{F}_k \right].$$