

Mathematical Foundations for Finance

Exercise sheet 8

Please upload your solutions until Wednesday, 17/11/2021, 12:00 using the link on the course website.

Exercise 8.1 Let $W = (W_t)_{t \geq 0}$ be a Brownian motion (BM) defined on some probability space (Ω, \mathcal{F}, P) (without filtration). Show that

- (a) $W^1 := -W$ is a BM.
- (b) $W_t^2 := W_{T+t} - W_T$, $t \geq 0$, is a BM for any $T \in (0, \infty)$.
- (c) $W^3 := \alpha B + \sqrt{1 - \alpha^2} B'$ is a BM, where B and B' are two independent BMs and $\alpha \in [0, 1]$.
- (d) Show that the independence of B and B' in (c) cannot be omitted, i.e., if B and B' are *not* independent, then W^3 need not be a BM. Give two examples.

Exercise 8.2 Let $W = (W_t)_{t \geq 0}$ be a Brownian motion. Let $a, b > 0$ and define

$$\begin{aligned}\tau_a &= \inf\{t \geq 0 \mid W_t > a\}, \\ \sigma_{a,b} &= \inf\{t \geq 0 \mid W_t > a + bt\}.\end{aligned}$$

- (a) Show that for $\tau \in \{\tau_a, \sigma_{a,b}\}$ and all $\alpha \in \mathbb{R}$ we have that

$$E \left[e^{\alpha W_{\tau \wedge t} - \frac{1}{2} \alpha^2 (\tau \wedge t)} \right] = 1.$$

- (b) Using your result from (a) show that

$$e^{\alpha a} E \left[e^{-\frac{1}{2} \alpha^2 \tau_a} \right] = 1,$$

and use this to conclude by an appropriate choice of α that the Laplace transform ϕ_{τ_a} of τ_a is given by

$$\phi_{\tau_a}(\lambda) := E \left[e^{-\lambda \tau_a} \right] = e^{-a\sqrt{2\lambda}}, \quad \lambda > 0.$$

Hint 1: Make use of dominated convergence theorem.

Hint 2: Use that $W_{\tau_a} = a$ P -a.s.; we will show this in another exercise sheet.

- (c) Using your result from (a) show that

$$e^{\alpha a} E \left[e^{(ab - \frac{1}{2} \alpha^2) \sigma_{a,b}} \right] = 1,$$

and use this to conclude by an appropriate choice of α that the Laplace transform $\phi_{\sigma_{a,b}}$ of $\sigma_{a,b}$ is given by

$$\phi_{\sigma_{a,b}}(\lambda) := E \left[e^{-\lambda \sigma_{a,b}} \right] = e^{-a(b + \sqrt{b^2 + 2\lambda})}, \quad \lambda > 0.$$

Hint 1: Make use of dominated convergence theorem.

Hint 2: Use that $W_{\sigma_{a,b}} = a + b\sigma_{a,b}$ P -a.s. on the event $\sigma_{a,b} < \infty$.

- (d) Show that τ_a is P -a.s. finite for any $a > 0$ and that $\sigma_{a,b}$ takes the value of $+\infty$ with a positive probability for any $a, b > 0$.

Exercise 8.3 Let $W = (W_t)_{t \geq 0}$ be a Brownian motion defined on some sufficiently rich filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying the usual conditions.

- (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary continuous convex function. Show that if the stochastic process $(f(W_t))_{t \geq 0}$ is integrable, then it is a (P, \mathbb{F}) -submartingale.

Hint: We have done something similar in discrete time.

- (b) Given a (P, \mathbb{F}) -martingale $(M_t)_{t \geq 0}$ and a measurable function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$, show that the process

$$(M_t + g(t))_{t \geq 0}$$

is a (P, \mathbb{F}) -supermartingale if and only if g is decreasing, and a (P, \mathbb{F}) -submartingale if and only if g is increasing.

- (c) Show that the following stochastic processes are (P, \mathbb{F}) -submartingales but not martingales:

(i) W^2 ,

(ii) $e^{\alpha W}$ for any $\alpha \in \mathbb{R}$.

Hint: Use the result from (a) and (b), respectively.

- (d) Show that any (P, \mathbb{F}) -local martingale which is null at 0 and uniformly bounded from below is a (P, \mathbb{F}) -supermartingale.

Hint: We have done this in discrete time already.