Mathematical Foundations for Finance

Exercise sheet 8

Please upload your solutions until Wednesday, 17/11/2021, 12:00 using the link on the course website.

Exercise 8.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion (BM) defined on some probability space $(\Omega, \mathcal{F}, P)$ (without filtration). Show that

(a) $W^1 := -W$ is a BM.

(b) $W^2_t := W_{T+t} - W_T$, $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(c) $W^3 := \alpha B + \sqrt{1-\alpha^2} B'$ is a BM, where B and B' are two independent BMs and $\alpha \in [0,1]$.

(d) Show that the independence of B and B' in (c) cannot be omitted, i.e., if B and B' are *not* independent, then W^3 need not be a BM. Give two examples.

Solution 8.1 We first recall the definition of a Brownian motion (without filtration) in order to know what needs to be checked. A *Brownian motion* with respect to P is a real-valued stochastic process $W = (W_t)_{t\geq 0}$ such that

(BM0) $W_0 = 0$ P-a.s.

(BM1′) For any $n \in \mathbb{N}$ and any times $0 = t_0 < t_1 < \cdots < t_n < \infty$, the increments $W_{t_i} - W_{t_{i-1}}$ are independent and normally distributed with variance $t_i - t_{i-1}$ under P, i.e.

$$W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1}) \text{ for } i = 1, \dots, n.$$

(BM2) W has P-a.s. continuous trajectories.

(a) We check (BM0), (BM1′) and (BM2) separately.

(BM0) This is clear since $W^1_0 = -W_0 = 0$ P-a.s.

(BM1′) Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then we have

$$W^1_{t_i} - W^1_{t_{i-1}} = -(W_{t_i} - W_{t_{i-1}}), \quad i = 1, \dots, n,$$

which are independent under P. Since $X \sim \mathcal{N}(0, \sigma^2)$ if and only if $-X \sim \mathcal{N}(0, \sigma^2)$, we also conclude that $W^1_{t_i} - W^1_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})$.

(BM2) This is trivial, since $W^1 = -W$. The sign does not alter continuity.

(b) We check (BM0), (BM1′) and (BM2) separately.

(BM0) We obviously have $W^2_0 = W_T - W_T = 0$ P-a.s.

(BM1′) Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then we have

$$\begin{aligned} W^2_{t_i} - W^2_{t_{i-1}} &= W_{T+t_i} - W_T - (W_{T+t_{i-1}} - W_T) \\ &= W_{T+t_i} - W_{T+t_{i-1}}, \quad i = 1, \dots, n. \end{aligned}$$

Denoting $t'_i = T + t_i$, we see from the definition (BM1′) that the increments of W^2 are independent under P, and since $t'_i - t'_{i-1} = t_i - t_{i-1}$, we also conclude that

$$W^2_{t_i} - W^2_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1}), \text{ for } i = 1, \dots, n.$$

(BM2) This is again easy, since W^2 is simply W shifted in time by T minus a random variable which does not depend on t.

(c) We check (BM0), (BM1′) and (BM2) separately.

(BM0) $W_0^3 = \alpha B_0 + \sqrt{1-\alpha^2} B_0' = 0$ P-a.s., since both B_0 and B_0' are equal to 0 P-a.s.

(BM1′) Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then we have

$$W_{t_i}^3 - W_{t_{i-1}}^3 = \alpha \left(B_{t_i} - B_{t_{i-1}}\right) + \sqrt{1-\alpha^2} \left(B_{t_i}' - B_{t_{i-1}}'\right), \quad i = 1, \dots, n.$$

Since B and B' are independent under P, we conclude that the right-hand side is an independent family of random variables. Since B and B' are BMs, we additionally have that

$$\begin{aligned} B_{t_i} - B_{t_{i-1}} &\sim \mathcal{N}(0, t_i - t_{i-1}), \quad i = 1, \dots, n, \\ B_{t_i}' - B_{t_{i-1}}' &\sim \mathcal{N}(0, t_i - t_{i-1}), \quad i = 1, \dots, n. \end{aligned}$$

Recall the general fact that if $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0, \eta^2)$ are independent, then we have for any linear combination $s_1 X + s_2 Y$ that

$$s_1 X + s_2 Y \sim \mathcal{N}(0, s_1^2 \sigma^2 + s_2^2 \eta^2).$$

Using this, we conclude that

$$\begin{aligned} \alpha \left(B_{t_i} - B_{t_{i-1}}\right) &+ \sqrt{1-\alpha^2} \left(B_{t_i}' - B_{t_{i-1}}'\right) \\ &\sim \mathcal{N}(0, \alpha^2 (t_i - t_{i-1}) + (1-\alpha^2)(t_i - t_{i-1})) \\ &= \mathcal{N}(0, t_i - t_{i-1}). \end{aligned}$$

(BM2) This is evident, since W^3 is a linear combination of two processes whose paths are P-a.s. continuous.

(d) Two possible choices are $B = \pm B'$. In this case we have

$$W^3 = (\alpha \pm \sqrt{1-\alpha^2}) B,$$

which is not a Brownian motion because $W_1^3 \sim \mathcal{N}(0, \left((\alpha \pm \sqrt{1-\alpha^2})\right)^2)$ and $(\alpha \pm \sqrt{1-\alpha^2}))^2 \neq 1$ in general.

Exercise 8.2 Let $W = (W_t)_{t \geq 0}$ be a Brownian motion. Let $a, b > 0$ and define

$$\begin{aligned} \tau_a &= \inf\{t \geq 0 \,|\, W_t > a\}, \\ \sigma_{a,b} &= \inf\{t \geq 0 \,|\, W_t > a + bt\}. \end{aligned}$$

(a) Show that for $\tau \in \{\tau_a, \sigma_{a,b}\}$ and all $\alpha \in \mathbb{R}$ we have that

$$E\left[e^{\alpha W_{\tau \wedge t} - \frac{1}{2}\alpha^2 (\tau \wedge t)}\right] = 1.$$

(b) Using your result from (a) show that

$$e^{\alpha a} E\left[e^{-\frac{1}{2}\alpha^2 \tau_a}\right] = 1,$$

and use this to conclude by an appropriate choice of α that the Laplace transform ϕ_{τ_a} of τ_a is given by

$$\phi_{\tau_a}(\lambda) := E\left[e^{-\lambda \tau_a}\right] = e^{-a\sqrt{2\lambda}}, \quad \lambda > 0.$$

Hint 1: Make use of dominated convergence theorem.
Hint 2: Use that $W_{\tau_a} = a$ P-a.s.; we will show this in another exercise sheet.

(c) Using your result from (a) show that

$$e^{\alpha a} E\left[e^{(ab-\frac{1}{2}\alpha^2)\sigma_{a,b}}\right] = 1,$$

and use this to conclude by an appropriate choice of α that the Laplace transform $\phi_{\sigma_{a,b}}$ of $\sigma_{a,b}$ is given by

$$\phi_{\sigma_{a,b}}(\lambda) := E\left[e^{-\lambda\sigma_{a,b}}\right] = e^{-a(b+\sqrt{b^2+2\lambda})}, \quad \lambda > 0.$$

Hint 1: Make use of dominated convergence theorem.
Hint 2: Use that $W_{\sigma_{a,b}} = a + b\sigma_{a,b}$ P-a.s. on the event $\sigma_{a,b} < \infty$.

(d) Show that τ_a is P-a.s. finite for any $a > 0$ and that $\sigma_{a,b}$ takes the value of $+\infty$ with a positive probability for any $a, b > 0$.

Solution 8.2

(a) We know from Proposition IV.2.2 in the lecture notes that the process $M = (M_t)_{t\geq 0}$ given by

$$M_t = e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$$

is a martingale for all $\alpha \in \mathbb{R}$. The stopping theorem (Theorem IV.2.1 in the lecture notes) then implies that for any stopping time τ, the stopped process M^τ is also a martingale, which gives that

$$1 = E[M_0] = E[M_0^\tau] = E[M_t^\tau] = E[M_{\tau\wedge t}] = E\left[e^{\alpha W_{\tau\wedge t} - \frac{1}{2}\alpha^2(\tau\wedge t)}\right],$$

since $\tau \wedge t$ is a bounded stopping time for all $t \geq 0$.

(b) We clearly have that $W_{\tau_a\wedge t} \leq a$ by the definition of τ_a. As a consequence, we have that $M_{\tau_a\wedge t} \leq e^{\alpha a}$ for all $t > 0$. Dominated convergence theorem therefore gives that

$$\begin{aligned} 1 &= \lim_{t\to\infty} E\left[e^{\alpha W_{\tau_a\wedge t} - \frac{1}{2}\alpha^2(\tau_a\wedge t)}\right] = E\left[\lim_{t\to\infty} e^{\alpha W_{\tau_a\wedge t} - \frac{1}{2}\alpha^2(\tau_a\wedge t)}\right] \\ &= E\left[e^{W_{\tau_a} - \frac{1}{2}\alpha^2\tau_a}\right] = e^{\alpha a} E\left[e^{-\frac{1}{2}\alpha^2\tau_a}\right], \end{aligned}$$

where the last equality follows from the fact that $W_{\tau_a} = a$ P-a.s. Reorganizing the above and setting $\alpha = \sqrt{2\lambda}$ for any $\lambda > 0$ gives that

$$E\left[e^{-\lambda\tau_a}\right] = e^{-a\sqrt{2\lambda}}, \quad \lambda > 0.$$

(c) We proceed analogously to (b). We have that $W_{\sigma_{a,b}\wedge t} \leq a + b(\sigma_{a,b}\wedge t)$ by the definition of $\sigma_{a,b}$. As a consequence, we have that

$$M_{\sigma_{a,b}\wedge t} \leq \exp\left(\alpha a + \left(\alpha b - \frac{1}{2}\alpha^2\right)(\sigma_{a,b}\wedge t)\right).$$

The right-hand side is not yet independent of t, but if we assume that

$$\alpha b < \frac{1}{2}\alpha^2 \quad \iff \quad \alpha > 2b,$$

then $M_{\sigma_{a,b}\wedge t} \leq e^{\alpha a}$. So we can again apply dominated convergence theorem and obtain

$$\begin{aligned} 1 &= \lim_{t\to\infty} E\left[e^{\alpha W_{\sigma_{a,b}\wedge t} - \frac{1}{2}\alpha^2(\sigma_{a,b}\wedge t)} \mathbb{1}_{\sigma_{a,b}<\infty}\right] = E\left[\lim_{t\to\infty} e^{\alpha W_{\sigma_{a,b}\wedge t} - \frac{1}{2}\alpha^2(\sigma_{a,b}\wedge t)} \mathbb{1}_{\sigma_{a,b}<\infty}\right] \\ &= E\left[e^{\alpha W_{\sigma_{a,b}} - \frac{1}{2}\alpha^2\sigma_{a,b}} \mathbb{1}_{\sigma_{a,b}<\infty}\right] = E\left[e^{\alpha(a+b\sigma_{a,b}) - \frac{1}{2}\alpha^2\sigma_{a,b}} \mathbb{1}_{\sigma_{a,b}<\infty}\right] = e^{\alpha a} E\left[e^{(\alpha b - \frac{1}{2}\alpha^2)\sigma_{a,b}} \mathbb{1}_{\sigma_{a,b}<\infty}\right], \end{aligned}$$

where the fourth equality uses that $W_{\sigma_{a,b}} = a + b\sigma_{a,b}$. Reorganizing the above and setting $\alpha = b + \sqrt{b^2 + 2\lambda} > 2b$ for any $\lambda > 0$ gives that

$$E\left[e^{-\lambda\sigma_{a,b}}\right] = E\left[e^{-\lambda\sigma_{a,b}} \mathbb{1}_{\sigma_{a,b}<\infty}\right] = e^{-a(b+\sqrt{b^2+2\lambda})}.$$

(d) We have for any stopping time τ and any $\lambda > 0$ that

$$E\left[e^{-\lambda\tau}\right] = E\left[e^{-\lambda\tau}\mathbb{1}_{\{\tau<\infty\}} + e^{-\lambda\tau}\mathbb{1}_{\{\tau=\infty\}}\right] = E\left[e^{-\lambda\tau}\mathbb{1}_{\{\tau<\infty\}}\right].$$

Since $e^{-\lambda\tau} \leq 1$ for all $\lambda > 0$, dominated convergence theorem gives that

$$\lim_{\lambda\downarrow 0} E\left[e^{-\lambda\tau}\mathbb{1}_{\{\tau<\infty\}}\right] = E\left[\lim_{\lambda\downarrow 0} e^{-\lambda\tau}\mathbb{1}_{\{\tau<\infty\}}\right] = E\left[\mathbb{1}_{\{\tau<\infty\}}\right] = P\left[\tau < \infty\right].$$

So we conclude that

$$\lim_{\lambda\downarrow 0} \phi_\tau(\lambda) = P\left[\tau < \infty\right].$$

Using the expressions derived for the Laplace transforms of τ_a and $\sigma_{a.b}$ from (b) and (c), we obtain

$$\begin{aligned} P\left[\tau_a < \infty\right] &= \lim_{\lambda\downarrow 0} e^{-a\sqrt{2\lambda}} = 1, \\ P\left[\sigma_{a,b} < \infty\right] &= \lim_{\lambda\downarrow 0} e^{-a(b+\sqrt{b^2+2\lambda})} = e^{-2ab} < 1. \end{aligned}$$

So while τ_a is P-a.s. finite for any $a > 0$, $\sigma_{a,b}$ takes the value of $+\infty$ with probability $1 - e^{-2ab}$.

Exercise 8.3 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some sufficiently rich filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous convex function. Show that if the stochastic process $\big(f(W_t)\big)_{t\geq 0}$ is integrable, then it is a $(P, \mathbb{F})$-submartingale.
Hint: We have done something similar in discrete time.

(b) Given a $(P, \mathbb{F})$-martingale $(M_t)_{t\geq 0}$ and a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$, show that the process

$$\big(M_t + g(t)\big)_{t\geq 0}$$

is a $(P, \mathbb{F})$-supermartingale if and only if g is decreasing, and a $(P, \mathbb{F})$-submartingale if and only if g is increasing.

(c) Show that the following stochastic processes are $(P, \mathbb{F})$-submartingales but not martingales:

(i) W^2,

(ii) $e^{\alpha W}$ for any $\alpha \in \mathbb{R}$.

Hint: Use the result from (a) and (b), respectively.

(d) Show that any $(P, \mathbb{F})$-local martingale which is null at 0 and uniformly bounded from below is a $(P, \mathbb{F})$-supermartingale.
Hint: We have done this in discrete time already.

Solution 8.3

(a) First recall that W is a $(P, \mathbb{F})$-martingale. Adaptedness is clear since f is assumed to be continuous. Integrability is assumed as well. Then by Jensen's inequality for conditional expectations, we can compute

$$E\left[f(W_t)\,|\,\mathcal{F}_s\right] \geq f\big(E\left[W_t\,|\,\mathcal{F}_s\right]\big) = f(W_s) \qquad P\text{-a.s.}$$

for all $t \geq s$, and thus conclude that $f(W)$ is a $(P, \mathbb{F})$-submartingale.

(b) For any measurable function $g : \mathbb{R}_+ \to \mathbb{R}$, we have that $M_t + g(t)$ is $\mathcal{F}_t$-measurable and

$$E[|M_t + g(t)|] \leq E[|M_t|] + E[|g(t)|] = E[|M_t|] + |g(t)| < \infty.$$

Hence $\big(M_t + g(t)\big)_{t\geq 0}$ is adapted and integrable. We can then compute

$$E[M_t + g(t) \,|\, \mathcal{F}_s] = E[M_t \,|\, \mathcal{F}_s] + g(t) = M_s + g(s) + g(t) - g(s) \qquad P\text{-a.s.}$$

for all $t \geq s$. As a result, $\big(M_t + g(t)\big)_{t\geq 0}$ has the $(P, \mathbb{F})$-supermartingale property, i.e.

$$E[M_t + g(t) \,|\, \mathcal{F}_s] \leq M_s + g(s) \qquad P\text{-a.s.}$$

for all $t > s$, if and only if g is decreasing. Analogously, $\big(M_t + g(t)\big)_{t\geq 0}$ has the $(P, \mathbb{F})$-submartingale property, i.e.

$$E[M_t + g(t) \,|\, \mathcal{F}_s] \geq M_s + g(s) \qquad P\text{-a.s.}$$

for all $t > s$, if and only if g is increasing.

(c) (i) Note that $W_t^2 = W_t^2 - t + g(t)$, where $g(t) := t$. By Proposition IV.2.2. in the lecture notes, we know that $\big(W_t^2 - t\big)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale; hence, using that g is increasing, by (b) we can conclude that W^2 is a $(P, \mathbb{F})$-submartingale.

In order to show that W^2 is not a martingale, we can use the martingale property of $\big(W_t^2 - t\big)_{t\geq 0}$ to compute

$$E\left[W_t^2 \,\middle|\, \mathcal{F}_s\right] = E\left[W_t^2 - t \,\middle|\, \mathcal{F}_s\right] + t = W_s^2 - s + t > W_s^2 \qquad P\text{-a.s.},$$

showing that W^2 is not a $(P, \mathbb{F})$-martingale.

Alternatively, by Jensen's inequality for conditional expectations we have that

$$E\left[W_t^2 \,\middle|\, \mathcal{F}_s\right] \geq (E[W_t \,|\, \mathcal{F}_s])^2 = W_s^2,$$

and the inequality is strict with positive probability because $x \mapsto x^2$ is strictly convex and W_t is not P-a.s. constant. So W^2 is a submartingale but not a martingale. The same argument can be used for (c) with $x \mapsto e^{\alpha x}$.

(ii) Adaptedness is clear since the transformation $x \mapsto e^{\alpha x}$ is continuous, and since we know that $W_t \overset{d}{=} W_t - W_0$ is $\mathcal{N}(0, t)$-distributed, the random variable $e^{\alpha W_t}$ is integrable. Noting that $x \mapsto e^{\alpha x}$ is also a convex function, we can then apply (a) to conclude that $e^{\alpha W}$ is a $(P, \mathbb{F})$-submartingale.

Next, Proposition IV.2.2. in the lecture notes gives us that $\big(e^{\alpha W_t - \frac{1}{2}\alpha^2 t}\big)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale; hence, we can compute

$$E\left[e^{\alpha W_t} \,\middle|\, \mathcal{F}_s\right] = E\left[e^{\alpha W_t - \frac{1}{2}\alpha^2 t} \,\middle|\, \mathcal{F}_s\right] e^{\frac{1}{2}\alpha^2 t} = e^{\alpha W_s} e^{\frac{1}{2}\alpha^2 (t-s)} > e^{\alpha W_s} \qquad P\text{-a.s.},$$

showing that $e^{\alpha W}$ is not a $(P, \mathbb{F})$-martingale.

(d) Let $(X_t)_{t\geq 0}$ be a $(P, \mathbb{F})$-local martingale null at 0 and uniformly bounded from below by $-a \leq 0$ and denote by $(\tau_n)_{n\in\mathbb{N}}$ a localizing sequence. Since $\lim_{n\to\infty} \tau_n = \infty$ P-a.s., we have

$$\lim_{n\to\infty} X_{t\wedge\tau_n} = X_t \qquad P\text{-a.s.}$$

Moreover, since $(X_t)_{t\geq 0}$ is uniformly bounded from below by $-a$, we have that $X_{t\wedge\tau_n} \geq -a$ and thus $0 \leq |X_{t\wedge\tau_n}| \leq X_{t\wedge\tau_n} + 2a$ for all $n \in \mathbb{N}$. By Fatou's lemma, we can then compute

$$E[|X_t|] = E\left[\lim_{n\to\infty} |X_{t\wedge\tau_n}|\right] \leq \liminf_{n\to\infty} E[|X_{t\wedge\tau_n}|] \leq \liminf_{n\to\infty} E[X_{t\wedge\tau_n}] + 2a = 2a < \infty,$$

where the last equality uses the martingale property of X^{τ_n} and the fact that it is null at 0. We have thus proved integrability. Since adaptedness is clear by the definition of a local martingale, it only remains to show the $(P, \mathbb{F})$-supermartingale property. Using again that $X_{t \wedge \tau_n} \geq -a$ for all $n \in \mathbb{N}$, we can apply Fatou's lemma to obtain for $t > s$

$$E\left[X_t \,|\, \mathcal{F}_s\right] = E\left[\lim_{n\to\infty} X_{t\wedge\tau_n} \,\middle|\, \mathcal{F}_s\right] \leq \liminf_{n\to\infty} E\left[X_{t\wedge\tau_n} \,|\, \mathcal{F}_s\right] = X_s,$$

as desired.