ETH Zürich, D-MATH
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# Non-Life Insurance: Mathematics and Statistics

## Exercise sheet 13

**Exercise 13.1 Chain-Ladder Algorithm**

We write $i = 1, \ldots, I$ for the accident years denoting the years of claims occurrence. For every accident year we consider development years $j = 0, \ldots, J$. For all $i = 1, \ldots, I$ and $j = 0, \ldots, J$ we write $C_{i,j}$ for the cumulative payments up to development year $j$ for all claims that have occurred in accident year $i$. For simplicity, we set $I = J + 1 = 10$. Assume that we have observations

$$\mathcal{D}_I = \{C_{i,j} \,|\, i + j \leq I, \, 1 \leq i \leq I, \, 0 \leq j \leq J\}$$

given by the following upper claims reserving triangle:

| accident | development year $j$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| year $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 5'946'975 | 9'668'212 | 10'563'929 | 10'771'690 | 10'978'394 | 11'040'518 | 11'106'331 | 11'121'181 | 11'132'310 | 11'148'124 |
| 2 | 6'346'756 | 9'593'162 | 10'316'383 | 10'468'180 | 10'536'004 | 10'572'608 | 10'625'360 | 10'636'546 | 10'648'192 | |
| 3 | 6'269'090 | 9'245'313 | 10'092'366 | 10'355'134 | 10'507'837 | 10'573'282 | 10'626'827 | 10'635'751 | | |
| 4 | 5'863'015 | 8'546'239 | 9'268'771 | 9'459'424 | 9'592'399 | 9'680'740 | 9'724'068 | | | |
| 5 | 5'778'885 | 8'524'114 | 9'178'009 | 9'451'404 | 9'681'692 | 9'786'916 | | | | |
| 6 | 6'184'793 | 9'013'132 | 9'585'897 | 9'830'796 | 9'935'753 | | | | | |
| 7 | 5'600'184 | 8'493'391 | 9'056'505 | 9'282'022 | | | | | | |
| 8 | 5'288'066 | 7'728'169 | 8'256'211 | | | | | | | |
| 9 | 5'290'793 | 7'648'729 | | | | | | | | |
| 10 | 5'675'568 | | | | | | | | | |

Table 1: Upper claims reserving triangle $\mathcal{D}_I$.

This data set can be downloaded from https://people.math.ethz.ch/~wueth/exercises2.html by clicking on "Data to the Examples".

(a) Use the chain-ladder (CL) method to predict the lower triangle

$$\mathcal{D}_I^c = \{C_{i,j} \,|\, i + j > I, \, 1 \leq i \leq I, \, 0 \leq j \leq J\}.$$

(b) Calculate the CL reserves $\widehat{\mathcal{R}}_i^{\mathrm{CL}}$ for all accident years $i = 1, \ldots, I$.

**Exercise 13.2 Bornhuetter-Ferguson Algorithm**

Consider the same setup as in Exercise 13.1. We assume that we have prior informations $\widehat{\mu}_1, \ldots, \widehat{\mu}_I$ for the expected ultimate claims $\mathbb{E}[C_{1,J}], \ldots, \mathbb{E}[C_{I,J}]$ given by

| accident year $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| prior information $\widehat{\mu}_i$ | 11'653'101 | 11'367'306 | 10'962'965 | 10'616'762 | 11'044'881 |
| accident year $i$ | 6 | 7 | 8 | 9 | 10 |
| prior information $\widehat{\mu}_i$ | 11'480'700 | 11'413'572 | 11'126'527 | 10'986'548 | 11'618'437 |

Table 2: Prior informations $\widehat{\mu}_1, \ldots, \widehat{\mu}_I$.

(a) Use the Bornhuetter-Ferguson (BF) method to calculate the BF reserves $\widehat{\mathcal{R}}_i^{\mathrm{BF}}$ for all accident years $i = 1, \ldots, I$.

(b) Explain why in this example we have $\widehat{\mathcal{R}}_i^{\mathrm{CL}} < \widehat{\mathcal{R}}_i^{\mathrm{BF}}$, for all accident years $i = 2, \ldots, I$, where $\widehat{\mathcal{R}}_i^{\mathrm{CL}}$ denotes the CL reserves for accident year $i$ calculated in Exercise 13.1.

**Exercise 13.3 Over-Dispersed Poisson Model**
Consider the same setup as in Exercise 13.1. This time we apply the over-dispersed Poisson (ODP) model. To this end, for all $i = 1, \ldots, I$ and $j = 0, \ldots, J$, we write $X_{i,j}$ for all payments done in development year $j$ for claims with accident year $i$. According to Model Assumptions 9.10 of the lecture notes (version of December 17, 2020), we assume that there exist positive parameters $\mu_1, \ldots, \mu_I$, $\gamma_0, \ldots, \gamma_J$ and $\phi$ such that all $X_{i,j}$ are independent (in $i$ and $j$) with

$$\frac{X_{i,j}}{\phi} \sim \mathrm{Poi}(\mu_i \gamma_j / \phi),$$

for all $i = 1, \ldots, I$ and $j = 0, \ldots, J$, and side constraint $\sum_{j=0}^{J} \gamma_j = 1$ holds.

(a) Determine the MLEs of $\mu_1, \ldots, \mu_I$ and $\gamma_0, \ldots, \gamma_J$.

(b) Calculate the ODP reserves $\widehat{\mathcal{R}}_i^{\mathrm{ODP}}$ for all accident years $i = 1, \ldots, I$. What do you observe?

(c) Perform a GLM analysis for the payments $X_{i,j}$ using the ODP model in order to check the results obtained in part (b).

**Exercise 13.4 Mack's Formula and Merz-Wüthrich (MW) Formula (R Exercise)**
Consider the same setup as in Exercise 13.1.

(a) Write an R code using the R package `ChainLadder` in order to determine the following quantities:

- the conditional mean square error of prediction

$$\mathrm{msep}^{\mathrm{Mack}}_{C_{i,J}|\mathcal{D}_I}\left(\widehat{C}_{i,J}^{\mathrm{CL}}\right),$$

given in formula (9.21) of the lecture notes, for all accident years $i = 1, \ldots, I$;

- the conditional mean square error of prediction for aggregated accident years

$$\mathrm{msep}^{\mathrm{Mack}}_{\sum_{i=1}^{I} C_{i,J}|\mathcal{D}_I}\left(\sum_{i=1}^{I}\widehat{C}_{i,J}^{\mathrm{CL}}\right),$$

given in formula (9.22) of the lecture notes;

- the one-year (run-off) uncertainty

$$\mathrm{msep}^{\mathrm{MW}}_{\mathrm{CDR}_{i,I+1}|\mathcal{D}_I}(0),$$

given in formula (9.34) of the lecture notes, for all accident years $i = 1, \ldots, I$;

- the one-year (run-off) uncertainty for aggregated accident years

$$\mathrm{msep}^{\mathrm{MW}}_{\sum_{i=1}^{I}\mathrm{CDR}_{i,I+1}|\mathcal{D}_I}(0),$$

given in formula (9.35) of the lecture notes.

The references for the four formulas above correspond to the version of the lecture notes of December 17, 2020.

(b) Interpret the square-rooted conditional mean square errors of prediction relative to the claims reserves calculated in Exercise 13.1.

(c) Interpret the square-rooted one-year (run-off) uncertainties relative to the square-rooted conditional mean square errors of prediction.