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Non-Life Insurance: Mathematics and Statistics

Exercise sheet 2

Exercise 2.1 Maximum Likelihood and Hypothesis Test
Let $Y_1, \ldots, Y_n$ be claim amounts in CHF that an insurance company has to pay. We assume that $Y_1, \ldots, Y_n$ are independent and identically distributed (i.i.d.) random variables following a log-normal distribution with unknown parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Then, by definition, $\log Y_1, \ldots, \log Y_n$ are i.i.d. Gaussian random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Let $n = 8$ and suppose that we have the following observations $x_1, \ldots, x_8$ for $\log Y_1, \ldots, \log Y_8$:

$$x_1 = 9, \quad x_2 = 4, \quad x_3 = 6, \quad x_4 = 7, \quad x_5 = 3, \quad x_6 = 11, \quad x_7 = 6, \quad x_8 = 10.$$

(a) Write down the joint density $f_{\mu,\sigma^2}(x_1, \ldots, x_8)$ of $\log Y_1, \ldots, \log Y_8$.

(b) Calculate $\log f_{\mu,\sigma^2}(x_1, \ldots, x_8)$.

(c) Calculate the maximum likelihood estimates (MLEs)

$$(\widehat{\mu}, \widehat{\sigma}^2) = \arg \max_{(\mu,\sigma^2) \in \mathbb{R} \times \mathbb{R}_{>0}} \log f_{\mu,\sigma^2}(x_1, \ldots, x_8).$$

(d) Now suppose that we are interested in the mean μ of the logarithms of the claim amounts. An expert claims that $\mu = 6$. Perform a statistical test to test the null hypothesis H_0: $\mu = 6$ against the (two-sided) alternative hypothesis H_1: $\mu \neq 6$.

Exercise 2.2 Chebychev's Inequality and Law of Large Numbers
Suppose that an insurance company provides insurance against bike theft. In our model a bike gets stolen with a probability of 0.1, and in case of a theft the insurance company has to pay 1'000 CHF. We assume that we have n i.i.d. risks $X_1, \ldots, X_n$ with

$$X_i = \begin{cases} 1'000, & \text{with probability } 0.1, \\ 0, & \text{with probability } 0.9, \end{cases}$$

for all $i = 1, \ldots, n$. In this exercise we are interested in the probability

$$p(n) \stackrel{\text{def}}{=} \mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq 0.1\mu\right]$$

of a deviation of the sample mean $\frac{1}{n}\sum_{i=1}^{n} X_i$ to the mean claim size $\mu = \mathbb{E}[X_1]$ of at least 10%, and how diversification effects this probability.

(a) Calculate μ.

(b) Suppose that $n = 1$. Calculate $p(1)$.

(c) Suppose that $n = 1'000$. Calculate $p(1'000)$.

(d) Apply Chebychev's inequality to derive a minimum number n of risks such that $p(n) < 0.01$.

(e) What can you say about $\lim_{n \to \infty} \frac{1}{n}\sum_{i=1}^{n} X_i$?

Exercise 2.3 Central Limit Theorem
Let n be the number of claims and $Y_1, \ldots, Y_n$ the corresponding claim sizes, where we assume that $Y_1, \ldots, Y_n$ are i.i.d. random variables with expectation $\mathbb{E}[Y_1] = \mu$ and coefficient of variation $\mathrm{Vco}(Y_1) = 4$.

(a) Use the Central Limit Theorem to determine an approximate minimum number of claims n^{CLT} such that

$$\mathbb{P}\left[\left|\frac{1}{n^{\mathrm{CLT}}}\sum_{i=1}^{n^{\mathrm{CLT}}} Y_i - \mu\right| < 0.01\mu\right] \geq 0.95,$$

i.e. with probability of at least 95% the deviation of the sample mean $\frac{1}{n}\sum_{i=1}^{n} Y_i$ from the mean claim size μ is less than 1%.

(b) Compare the resulting minimum number of claims n^{CLT} to the corresponding minimum number of claims n^{Che} when using Chebychev's inequality instead of the Central Limit Theorem in part (a). What do you observe?

Exercise 2.4 Conditional Distribution and Variance Decomposition
Suppose that Θ follows an exponential distribution with parameter $\lambda > 0$. We assume that, conditionally given Θ, the number of claims N in a particular line of business of an insurance company is modeled by a Poisson distribution with frequency parameter Θv, where $v > 0$ denotes the volume, i.e. we have

$$\mathbb{P}[N = k|\Theta] = \begin{cases} e^{-\Theta v}\frac{(\Theta v)^k}{k!}, & \text{if } k \in \mathbb{N}, \\ 0, & \text{else.} \end{cases}$$

We remark that the expectation and the variance of a Poisson distribution are equal to its frequency parameter, i.e. here we have $\mathbb{E}[N|\Theta] = \mathrm{Var}(N|\Theta) = \Theta v$.

(a) Calculate $\mathbb{P}[N = 0]$.

(b) Calculate $\mathbb{E}[N]$.

(c) Show that $\mathrm{Var}(N) = \mathbb{E}[\mathrm{Var}(N|\Theta)] + \mathrm{Var}(\mathbb{E}[N|\Theta])$ and use this result to calculate $\mathrm{Var}(N)$.