# Non-Life Insurance: Mathematics and Statistics

## Exercise sheet 9

**Exercise 9.1 Value-at-Risk and Expected Shortfall**
Suppose that for the yearly claim amount $S$ of an insurance company in a given line of business we have $S \sim \text{LN}(\mu, \sigma^2)$ with $\mu = 20$ and $\sigma^2 = 0.015$. Moreover, we set the cost-of-capital rate $r_{\text{CoC}} = 6\%$. Then, the premium $\pi_{\text{CoC}}$ for the considered line of business using the cost-of-capital pricing principle with risk measure $\rho$ is given by

$$\pi_{\text{CoC}} = \mathbb{E}[S] + r_{\text{CoC}} \cdot \rho(S - \mathbb{E}[S]).$$

(a) Calculate $\pi_{\text{CoC}}$ using the value-at-risk (VaR) risk measure at security level $1 - q = 99.5\%$.

(b) Calculate $\pi_{\text{CoC}}$ using the expected shortfall risk measure at security level $1 - q = 99\%$.

(c) Which security level is needed such that $\pi_{\text{CoC}}$ using the VaR risk measure is equal to the price calculated in (b)?

(d) Let $U$ and $V$ be two independent copies of $\log S$. Show that on the one hand

$$\text{VaR}_{1-q}(U + V) > \text{VaR}_{1-q}(U) + \text{VaR}_{1-q}(V)$$

for all $1 - q \in \left(0, \frac{1}{2}\right)$, but on the other hand

$$\text{VaR}_{1-q}(U + V) < \text{VaR}_{1-q}(U) + \text{VaR}_{1-q}(V)$$

for all $1 - q \in \left(\frac{1}{2}, 1\right)$. In particular, the VaR is not subadditive, and hence not coherent.

**Exercise 9.2 Variance Loading Principle**
We would like to insure the car fleet given in Table 2 under the assumption that the total claim amounts for passenger cars, delivery vans and trucks can be modeled by independent compound Poisson distributions.

| $i$ | $v_i$ | $\lambda_i$ | $\mathbb{E}[Y_1^{(i)}]$ | $\text{Vco}(Y_1^{(i)})$ |
|---|---|---|---|---|
| passenger car | 40 | 25% | 2'000 | 2.5 |
| delivery van | 30 | 23% | 1'700 | 2.0 |
| truck | 10 | 19% | 4'000 | 3.0 |

Table 1: Volumes, claim frequencies, expected claim sizes and coefficients of variation of the claim sizes for the three sections of the car fleet.

(a) Calculate the expected claim amount of the car fleet.

(b) Calculate the premium for the car fleet using the variance loading principle with $\alpha = 3 \cdot 10^{-6}$.

**Exercise 9.3 Utility Indifference Price**
In this exercise we calculate the premium for the accident insurance of a given company COMP

| Group $i$ | $v_i$ | $\lambda_i$ | $Y_1^{(i)}$ |
|-----------|-------|-------------|-------------|
| 1 | 2'000 | 50% | $\Gamma(\gamma = 20, c = 0.01)$ |
| 2 | 10'000 | 10% | $\text{expo}(\kappa = 0.005)$ |

Table 2: Volumes, claim frequencies and claim size distributions for the two groups of employees.

using the utility indifference price principle. We suppose that all employees of COMP have been divided into two groups, depending on their work, and that the total claim amounts $S_1$ and $S_2$ of the two groups are independent and compound Poisson distributed with volumes, claim frequencies and claim size distributions as given in Table 1.

We write $S = S_1 + S_2$ for the total claim amount of COMP. Let $c_0$ be the initial capital of the insurance company that sells accident insurance to COMP.

(a) Let $u$ be a risk-averse utility function. Show that if the utility indifference price $\pi = \pi(u, S, c_0)$ exists, then it is unique and satisfies $\pi > \mathbb{E}[S]$.

(b) Calculate $\mathbb{E}[S]$.

(c) Calculate $\pi$ using the utility indifference price principle for the exponential utility function with parameter $\alpha = 1.5 \cdot 10^{-6}$.

(d) What happens to $\pi$ if we replace the compound Poisson distributions of $S_1$ and $S_2$ by Gaussian distributions with the same corresponding first two moments?

(e) For this part we assume that $S$ has a general compound Poisson distribution with expected number of claims $\lambda v \in \mathbb{N}$ and i.i.d. claim sizes $(Y_i)_{i \geq 1}$ for which the moment generating function $M_{Y_1}$ exists at $\alpha$ for a given $\alpha > 0$. Moreover, let $u$ be the exponential utility function with parameter $\alpha$, and $c_0 > 0$ a given initial capital. We write $\pi = \pi(u, S, c_0)$ for the utility indifference price for $S$. Now define

$$\widetilde{S} = \sum_{i=1}^{\lambda v} Y_i,$$

i.e. for $\widetilde{S}$ the number of claims is exactly given by $\lambda v$. Calculate the utility indifference price $\widetilde{\pi} = \widetilde{\pi}(u, \widetilde{S}, c_0)$ for $\widetilde{S}$ and compare $\widetilde{\pi}$ to $\pi$.

**Exercise 9.4  Esscher Premium**

Let $S$ be a random variable with distribution function $F$ and moment generating function $M_S$. Assume that there exists $r_0 > 0$ such that $M_S(r) < \infty$ for all $r \in (-r_0, r_0)$. For $\alpha \in (0, r_0)$, let $\pi_\alpha$ denote the Esscher premium of $S$.

(a) Show that if $S$ is non-deterministic, then $\pi_\alpha$ is strictly increasing in $\alpha$.

(b) Show that the Esscher premium for small values of $\alpha$ boils down to a variance loading principle.

(c) Suppose that $S \sim \text{CompPoi}(\lambda v, G)$, where $\lambda v > 0$ and $G$ is the distribution function of a gamma distribution with shape parameter $\gamma > 0$ and scale parameter $c > 0$. For which values of $\alpha$ does $\pi_\alpha$ exist? Calculate $\pi_\alpha$ where it is defined.