ETH Zürich, D-MATH
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Non-Life Insurance: Mathematics and Statistics

Solution sheet 10

Solution 10.1 Log-Linear Gaussian Regression Model

(a) In the log-linear Gaussian regression model we work with a stochastic model for the claim amounts $S_{i,j}$ for the risk classes $(i,j), 1 \leq i \leq 3, 1 \leq j \leq 4$, given in Table 1 on the exercise sheet. We assume that

$$X_{i,j} \stackrel{\text{def}}{=} \log \frac{S_{i,j}}{v_{i,j}} = \log S_{i,j} \sim \mathcal{N}(\beta_0 + \beta_{1,i} + \beta_{2,j}, \sigma^2),$$

where $\beta_0, \beta_{1,i}, \beta_{2,j} \in \mathbb{R}$ and $\sigma^2 > 0$, for all risk classes $(i,j), 1 \leq i \leq 3, 1 \leq j \leq 4$. The risk characteristics of the two tariff criteria vehicle type and driver age are now given by

$$\beta_{1,1} \text{ (passenger car)}, \quad \beta_{1,2} \text{ (delivery van)} \quad \text{and} \quad \beta_{1,3} \text{ (truck)},$$

and

$$\beta_{2,1} \text{ (21 - 30 years)}, \quad \beta_{2,2} \text{ (31 - 40 years)}, \quad \beta_{2,3} \text{ (41 - 50 years)} \quad \text{and} \quad \beta_{2,4} \text{ (51 - 60 years)}.$$

In order to get a unique solution, we set $\beta_{1,1} = \beta_{2,1} = 0$. Simplifying notation, we write $\boldsymbol{X} = (X_1, \ldots, X_M)'$ with $M = 12$ and

$$X_1 = X_{1,1}, \quad X_2 = X_{1,2}, \quad X_3 = X_{1,3}, \quad X_4 = X_{1,4}, \quad X_5 = X_{2,1}, \quad X_6 = X_{2,2},$$
$$X_7 = X_{2,3}, \quad X_8 = X_{2,4}, \quad X_9 = X_{3,1}, \quad X_{10} = X_{3,2}, \quad X_{11} = X_{3,3}, \quad X_{12} = X_{3,4}.$$

Moreover, we define

$$\boldsymbol{\beta} = (\beta_0, \beta_{1,2}, \beta_{1,3}, \beta_{2,2}, \beta_{2,3}, \beta_{2,4})' \in \mathbb{R}^{r+1},$$

with $r = 5$. Then, we assume that $\boldsymbol{X}$ has a multivariate Gaussian distribution

$$\boldsymbol{X} \sim \mathcal{N}(Z\boldsymbol{\beta}, \sigma^2 I),$$

where $I \in \mathbb{R}^{M \times M}$ denotes the identity matrix and $Z \in \mathbb{R}^{M \times (r+1)}$ is the so-called design matrix that satisfies

$$\mathbb{E}[\boldsymbol{X}] = Z\boldsymbol{\beta}.$$

For example for $m = 1$ we have

$$\mathbb{E}[X_m] = \mathbb{E}[X_1] = \mathbb{E}[X_{1,1}] = \beta_0 + \beta_{1,1} + \beta_{2,1} = \beta_0 = (1,0,0,0,0,0)\,\boldsymbol{\beta},$$

and for $m = 8$

$$\mathbb{E}[X_m] = \mathbb{E}[X_8] = \mathbb{E}[X_{2,4}] = \beta_0 + \beta_{1,2} + \beta_{2,4} = (1,1,0,0,0,1)\,\boldsymbol{\beta}.$$

Doing this for all $m \in \{1, \dots, 12\}$, we find the design matrix Z given in Table 1. Note that we can also let R find the design matrix by itself, see Listing 1 given below.

intercept (β_0)	van ($\beta_{1,2}$)	truck ($\beta_{1,3}$)	31-40y ($\beta_{2,2}$)	41-50y ($\beta_{2,3}$)	51-60y ($\beta_{2,4}$)
1	0	0	0	0	0
1	0	0	1	0	0
1	0	0	0	1	0
1	0	0	0	0	1
1	1	0	0	0	0
1	1	0	1	0	0
1	1	0	0	1	0
1	1	0	0	0	1
1	0	1	0	0	0
1	0	1	1	0	0
1	0	1	0	1	0
1	0	1	0	0	1

Table 1: Design matrix Z ($\beta_{1,1} = \beta_{2,1} = 0$).

(b) The R code used for parts (b), (c) and (d) is given in Listing 1 below. According to formula (7.11) of the lecture notes (version of December 17, 2020), the MLE of the parameter vector $\boldsymbol{\beta}$ is given by

$$\widehat{\boldsymbol{\beta}}^{\mathrm{MLE}} = [Z'(\sigma^2 I)^{-1} Z]^{-1} Z'(\sigma^2 I)^{-1} \boldsymbol{X} = (Z'Z)^{-1} Z' \boldsymbol{X}.$$

Note that $\widehat{\boldsymbol{\beta}}^{\mathrm{MLE}}$ does not depend on σ^2. Moreover, the design matrix Z has full column rank and, thus, $Z'Z$ is indeed invertible. We get the following tariff structure:

$\widehat{\beta}_0 = 7.688$	21-30y	31-40y	41-50y	51-60y	$\widehat{\beta}_{1,i}$
passenger car	2'182	1'759	1'500	1'501	0
delivery van	2'063	1'663	1'417	1'419	-0.056
truck	2'444	1'970	1'680	1'682	0.113
$\widehat{\beta}_{2,j}$	0	-0.216	-0.375	-0.374	

Table 2: Tariff structure resulting from the log-linear Gaussian regression model.

If we use the same parametrization as in Exercises 10.1 and 10.2, we get the following table:

$\exp\{\widehat{\beta}_0\} = 1$	21-30y	31-40y	41-50y	51-60y	$\exp\{\widehat{\beta}_{1,i}\}$
passenger car	2'182	1'759	1'500	1'501	1
delivery van	2'063	1'663	1'417	1'419	0.95
truck	2'444	1'970	1'680	1'682	1.12
$\exp\{\widehat{\beta}_{2,j}\}$	2'182	1'759	1'500	1'501	

Table 3: Tariff structure with the same parametrization as in Exercises 10.1 and 10.2.

Note that the tariffs in Tables 2 and 3 do not change with the different parametrization.

According to the R output, we get the following p-values for the individual parameters:

For every parameter, R calculates the corresponding p-value by applying a t-test to the null hypothesis that the parameter under consideration is equal to 0. While the p-values for $\widehat{\beta}_0, \widehat{\beta}_{1,3}, \widehat{\beta}_{2,2}, \widehat{\beta}_{2,3}, \widehat{\beta}_{2,4}$ are smaller than 0.05 and, thus, these parameters are significantly different from zero, the p-value for $\widehat{\beta}_{1,2}$ (delivery van) is fairly high. Hence, we might question

	$\widehat{\beta}_0$	$\widehat{\beta}_{1,2}$	$\widehat{\beta}_{1,3}$	$\widehat{\beta}_{2,2}$	$\widehat{\beta}_{2,3}$	$\widehat{\beta}_{2,4}$
p-value	≈ 0	0.2322	0.0366	0.0045	0.0003	0.0003

Table 4: Resulting p-values for the individual parameters.

if we really need the class delivery van. This is in line with the observations that the risk characteristics for the classes passenger car and delivery van are close to each other, see Tables 1, 2, 2 and 3.

(c) In order to check whether there is statistical evidence that the classification into different types of vehicles could be omitted, we define the null hypothesis of the reduced model:

$$H_0 : \beta_{1,2} = \beta_{1,3} = 0,$$

i.e. we set $p = 2$ parameters equal to 0. We can perform the same analysis as above to get the MLE $\widehat{\boldsymbol{\beta}}^{\text{MLE}}_{H_0}$ of the reduced model H_0. In particular, let Z_{H_0} be the design matrix Z without the second column van ($\beta_{1,2}$) and the third column truck ($\beta_{1,3}$). Then, we have

$$\widehat{\boldsymbol{\beta}}^{\text{MLE}}_{H_0} = (Z'_{H_0} Z_{H_0})^{-1} Z'_{H_0} \boldsymbol{X}.$$

Now, for all $m \in \{1, \dots, 12\}$ we define the fitted value $\widehat{X}^{\text{full}}_m$ of the full model and the fitted value $\widehat{X}^{H_0}_m$ of the reduced model. In particular, we have

$$\widehat{X}^{\text{full}}_m = \left[Z \widehat{\boldsymbol{\beta}}^{\text{MLE}} \right]_m$$

and

$$\widehat{X}^{H_0}_m = \left[Z_{H_0} \widehat{\boldsymbol{\beta}}^{\text{MLE}}_{H_0} \right]_m ,$$

where $[\cdot]_m$ denotes the m-th element of the corresponding vector, for all $m \in \{1, \dots, 12\}$. Moreover, we define the residual differences

$$SS^{\text{full}}_{\text{err}} = \sum_{m=1}^{M} \left(X_m - \widehat{X}^{\text{full}}_m \right)^2$$

and

$$SS^{H_0}_{\text{err}} = \sum_{m=1}^{M} \left(X_m - \widehat{X}^{H_0}_m \right)^2 .$$

According to formula (7.17) of the lecture notes (version of December 17, 2020), the test statistic

$$T = \frac{SS^{H_0}_{\text{err}} - SS^{\text{full}}_{\text{err}}}{SS^{\text{full}}_{\text{err}}} \frac{M - r - 1}{p} = 3 \, \frac{SS^{H_0}_{\text{err}} - SS^{\text{full}}_{\text{err}}}{SS^{\text{full}}_{\text{err}}}$$

has an F-distribution with degrees of freedom given by $\text{df}_1 = p = 2$ and $\text{df}_2 = M - r - 1 = 6$. We get

$$T \approx 8.336,$$

which corresponds to a p-value of approximately 1.85%. Thus, we can reject H_0 at significance level of 5%, i.e. there seems to be no statistical evidence that the classification into different types of vehicles could be omitted.

Listing 1: R code for Exercise 10.1.

```
### Load the observed claim amounts into a matrix
S <- matrix(c(2000,2200,2500,1800,1600,2000,1500,1400,1700,1600,1400,1600), nrow=3)

### Define the design matrix Z
Z <- matrix(c(rep(1,12),rep(0,4),rep(1,4),rep(0,12),rep(1,4),rep(c(0,1,0,0),3),
              rep(c(0,0,1,0),3),rep(c(0,0,0,1),3)), nrow=12)

### Store design matrix Z and log(S_{i,j}) in one dataset
data <- as.data.frame(cbind(Z[,-1],matrix(log(t(S)),nrow=12)))
colnames(data) <- c("van", "truck", "X31_40y", "X41_50y", "X51_60y", "observation")

### Apply the regression model
linear.model1 <- lm(formula = observation ~ van+truck+X31_40y+X41_50y+X51_60y, data=data)
summary(linear.model1)

### Fitted values
matrix(exp(fitted(linear.model1)), byrow=TRUE, nrow=3)

### We can also get the parameters by applying formula (7.11) of the lecture notes
solve(t(Z)%*%Z) %*% t(Z) %*% matrix(log(t(S)), nrow=12)

### We can also use R directly on the data (it finds the design matrix internally)
car <- c("passenger car", "van", "truck")
age <- c("X21_30y", "X31_40y", "X41_50y", "X51_60y")
dat <- expand.grid(car, age)
colnames(dat) <- c("car","age")
dat$observation <- as.vector(log(S))
linear.model1.direct <- lm(formula = observation ~ car+age, data=dat)
summary(linear.model1.direct)

### Apply the regression model under H_0, calculate the test statistic F and the p-value
linear.model2 <- lm(formula = observation ~ X31_40y+X41_50y+X51_60y, data=data)
test.stat <- 3*(sum((data[,6]-fitted(linear.model2))^2)-sum((data[,6]-fitted(linear.model1))^2))
                /sum((data[,6]-fitted(linear.model1))^2)
pf(test.stat, 2, 6, lower.tail=FALSE)

### We can also directly use anova to test H_0
anova(linear.model1,linear.model2)
```

Solution 10.2 Method of Bailey & Simon

In this exercise we work with two tariff criteria. The first criterion (vehicle type) has $I = 3$ risk characteristics:

$$\chi_{1,1} \text{ (passenger car)}, \quad \chi_{1,2} \text{ (delivery van)} \quad \text{and} \quad \chi_{1,3} \text{ (truck)}.$$

The second criterion (driver age) has $J = 4$ risk characteristics:

$$\chi_{2,1} \text{ (21 - 30 years)}, \quad \chi_{2,2} \text{ (31 - 40 years)}, \quad \chi_{2,3} \text{ (41 - 50 years)} \quad \text{and} \quad \chi_{2,4} \text{ (51 - 60 years)}.$$

The claim amounts $S_{i,j}$ for the risk classes $(i,j), 1 \le i \le 3, 1 \le j \le 4$, are given in Table 1 on the exercise sheet. The multiplicative tariff structure leads to the model

$$\mathbb{E}[S_{i,j}] = v_{i,j}\, \mu\, \chi_{1,i}\, \chi_{2,j},$$

for all $1 \le i \le 3, 1 \le j \le 4$, where we set the number of policies $v_{i,j} = 1$. Moreover, in order to get a unique solution, we set $\mu = 1$ and $\chi_{1,1} = 1$. Therefore, there remains to find the risk characteristics $\chi_{1,2}, \chi_{1,3}, \chi_{2,1}, \chi_{2,2}, \chi_{2,3}, \chi_{2,4}$. Using the method of Bailey & Simon, these risk characteristics are found by minimizing

$$X^2 = \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{(S_{i,j} - v_{i,j}\, \mu\, \chi_{1,i}\, \chi_{2,j})^2}{v_{i,j}\, \mu\, \chi_{1,i}\, \chi_{2,j}} = \sum_{i=1}^{3} \sum_{j=1}^{4} \frac{(S_{i,j} - \chi_{1,i}\, \chi_{2,j})^2}{\chi_{1,i}\, \chi_{2,j}}.$$

Let $i \in \{2,3\}$ (recall that we set $\widehat{\chi}_{1,1} = 1$). Then, $\widehat{\chi}_{1,i}$ is found by the solution of

$$\begin{aligned}
0 \stackrel{!}{=} \frac{\partial}{\partial \chi_{1,i}} X^2 &= \sum_{j=1}^{4} \frac{\partial}{\partial \chi_{1,i}} \frac{(S_{i,j} - \chi_{1,i}\,\chi_{2,j})^2}{\chi_{1,i}\,\chi_{2,j}} \\
&= \sum_{j=1}^{4} \frac{-2(S_{i,j} - \chi_{1,i}\,\chi_{2,j})\chi_{1,i}\,\chi_{2,j} - (S_{i,j} - \chi_{1,i}\,\chi_{2,j})^2}{\chi_{1,i}^2\,\chi_{2,j}} \\
&= \sum_{j=1}^{4} \frac{-2S_{i,j}\chi_{1,i}\,\chi_{2,j} + 2\chi_{1,i}^2\,\chi_{2,j}^2 - S_{i,j}^2 + 2S_{i,j}\chi_{1,i}\,\chi_{2,j} - \chi_{1,i}^2\,\chi_{2,j}^2}{\chi_{1,i}^2\,\chi_{2,j}} \\
&= \sum_{j=1}^{4} \frac{\chi_{1,i}^2\,\chi_{2,j}^2 - S_{i,j}^2}{\chi_{1,i}^2\,\chi_{2,j}} \\
&= \sum_{j=1}^{4} \chi_{2,j} - \frac{1}{\chi_{1,i}^2} \sum_{j=1}^{4} \frac{S_{i,j}^2}{\chi_{2,j}}.
\end{aligned}$$

Thus, for $i \in \{2,3\}$ we get

$$\widehat{\chi}_{1,i} = \left(\frac{\sum_{j=1}^{4} S_{i,j}^2 / \widehat{\chi}_{2,j}}{\sum_{j=1}^{4} \widehat{\chi}_{2,j}} \right)^{1/2}.$$

By an analogous calculation, one finds

$$\widehat{\chi}_{2,j} = \left(\frac{\sum_{i=1}^{3} S_{i,j}^2 / \widehat{\chi}_{1,i}}{\sum_{i=1}^{3} \widehat{\chi}_{1,i}} \right)^{1/2},$$

for $j \in \{1,2,3,4\}$. For solving these equations, one has to apply a root-finding algorithm like for example the Newton-Raphson method. We get the following multiplicative tariff structure:

	21-30y	31-40y	41-50y	51-60y	$\widehat{\chi}_{1,i}$
passenger car	2'176	1'751	1'491	1'493	1
delivery van	2'079	1'674	1'425	1'427	0.96
truck	2'456	1'977	1'684	1'686	1.13
$\widehat{\chi}_{2,j}$	2'176	1'751	1'491	1'493	

Table 5: Tariff structure resulting from the method of Bailey & Simon.

We see that the risk characteristics for the classes passenger car and delivery van are close to each other, whereas for trucks we have a higher tariff. Moreover, an insured with age between 21 and 30 years gets a considerably higher tariff than an insured with a higher age. The smallest tariff is assigned to insureds with age between 41 and 60 years. Note that we have

$$\sum_{i=1}^{3} \sum_{j=1}^{4} v_{i,j}\, \mu\, \widehat{\chi}_{1,i}\, \widehat{\chi}_{2,j} \;=\; 21'320 \;>\; 21'300 \;=\; \sum_{i=1}^{3} \sum_{j=1}^{4} S_{i,j},$$

which confirms the (systematic) positive bias of the method of Bailey & Simon shown in Lemma 7.2 of the lecture notes (version of December 17, 2020).

Solution 10.3 Method of Bailey & Jung

We use the same setup and the same notation as in the solution of Exercise 10.2. In order to get a unique solution, we again set $\mu = 1$ and $\chi_{1,1} = 1$. Using the method of Bailey & Jung, which is also called method of total marginal sums, the risk characteristics $\chi_{1,2}, \chi_{1,3}, \chi_{2,1}, \chi_{2,2}, \chi_{2,3}, \chi_{2,4}$ are found by solving the equations

$$\sum_{j=1}^{J} v_{i,j}\,\mu\,\chi_{1,i}\,\chi_{2,j} \;=\; \sum_{j=1}^{J} S_{i,j}, \qquad i \in \{2,3\},$$
$$\sum_{i=1}^{I} v_{i,j}\,\mu\,\chi_{1,i}\,\chi_{2,j} \;=\; \sum_{i=1}^{I} S_{i,j}, \qquad j \in \{1,2,3,4\}.$$

Since $I = 3, J = 4$ and we work with $v_{i,j} = 1$ and set $\mu = 1$, we get the equations

$$\sum_{j=1}^{4} \chi_{1,i}\,\chi_{2,j} \;=\; \sum_{j=1}^{4} S_{i,j}, \qquad i \in \{2,3\},$$
$$\sum_{i=1}^{3} \chi_{1,i}\,\chi_{2,j} \;=\; \sum_{i=1}^{3} S_{i,j}, \qquad j \in \{1,2,3,4\}.$$

Thus, for $i \in \{2,3\}$ (recall that we set $\widehat{\chi}_{1,1} = 1$) and $j \in \{1,2,3,4\}$, we get

$$\widehat{\chi}_{1,i} \;=\; \sum_{j=1}^{4} S_{i,j} \Big/ \sum_{j=1}^{4} \widehat{\chi}_{2,j}, \qquad \text{and,}$$
$$\widehat{\chi}_{2,j} \;=\; \sum_{i=1}^{3} S_{i,j} \Big/ \sum_{i=1}^{3} \widehat{\chi}_{1,i}.$$

Analogously to the method of Bailey & Simon, one has to solve this system of equations using a root-finding algorithm.
We get the following multiplicative tariff structure:

	21-30y	31-40y	41-50y	51-60y	$\widehat{\chi}_{1,i}$
passenger car	2'170	1'749	1'490	1'490	1
delivery van	2'076	1'673	1'425	1'425	0.96
truck	2'454	1'977	1'685	1'685	1.13
$\widehat{\chi}_{2,j}$	2'170	1'749	1'490	1'490	

Table 6: Tariff structure resulting from the method of Bailey & Jung.

We see that the results are very close to those in Exercise 10.2, where we applied the method of Bailey & Simon. However, now we have

$$\sum_{i=1}^{3}\sum_{j=1}^{4} v_{i,j}\,\mu\,\widehat{\chi}_{1,i}\,\widehat{\chi}_{2,j} \;=\; 21\text{'}300 \;=\; \sum_{i=1}^{3}\sum_{j=1}^{4} S_{i,j},$$

which comes as no surprise as we fitted the risk characteristics such that the above equality holds true.

Solution 10.4 Tweedie's Compound Poisson Model

(a) We can write S as

$$S \;=\; \sum_{i=1}^{N} Y_i,$$

where $N \sim \mathrm{Poi}(\lambda v)$, $Y_1, Y_2, \ldots \overset{\text{i.i.d.}}{\sim} G$ and N and $(Y_1, Y_2, \ldots)$ are independent. Since G is the distribution function of a gamma distribution, we have $G(0) = 0$ and, thus,

$$\mathbb{P}[S = 0] \;=\; \mathbb{P}[N = 0] \;=\; \exp\{-\lambda v\}.$$

Let $x \in (0, \infty)$. Then, the density f_S of S at x can be calculated as

$$f_S(x) \;=\; \frac{d}{dx}\,\mathbb{P}[S \le x],$$

where we have

$$\begin{aligned}
\mathbb{P}[S \le x] \;&=\; \sum_{n=0}^{\infty} \mathbb{P}[S \le x, N = n] \;=\; \sum_{n=0}^{\infty} \mathbb{P}[S \le x \,|\, N = n]\,\mathbb{P}[N = n] \\
&=\; \mathbb{P}[S \le x \,|\, N = 0]\,\mathbb{P}[N = 0] + \sum_{n=1}^{\infty} \mathbb{P}[S \le x \,|\, N = n]\,\mathbb{P}[N = n] \\
&=\; \mathbb{P}[N = 0] + \sum_{n=1}^{\infty} \mathbb{P}\left[\sum_{i=1}^{n} Y_i \le x\right] \mathbb{P}[N = n].
\end{aligned}$$

Since $Y_1, Y_2, \ldots \overset{\text{i.i.d.}}{\sim} \Gamma(\gamma, c)$, we get

$$\sum_{i=1}^{n} Y_i \;\sim\; \Gamma(n\gamma, c).$$

By writing f_n for the density function of $\Gamma(n\gamma, c)$, for all $n \in \mathbb{N}$, we get

$$\begin{aligned}
f_S(x) \;&=\; \frac{d}{dx}\left(\mathbb{P}[N = 0] + \sum_{n=1}^{\infty} \mathbb{P}\left[\sum_{i=1}^{n} Y_i \le x\right] \mathbb{P}[N = n]\right) \;=\; \sum_{n=1}^{\infty} \frac{d}{dx}\,\mathbb{P}\left[\sum_{i=1}^{n} Y_i \le x\right] \mathbb{P}[N = n] \\
&=\; \sum_{n=1}^{\infty} f_n(x)\,\mathbb{P}[N = n] \;=\; \sum_{n=1}^{\infty} \frac{c^{n\gamma}}{\Gamma(n\gamma)} x^{n\gamma - 1} \exp\{-cx\} \exp\{-\lambda v\} \frac{(\lambda v)^n}{n!} \\
&=\; \exp\{-(cx + \lambda v)\} \sum_{n=1}^{\infty} (\lambda v c^{\gamma})^n \frac{1}{\Gamma(n\gamma) n!} x^{n\gamma - 1} \\
&=\; \exp\left\{-(cx + \lambda v) + \log\left[\sum_{n=1}^{\infty} (\lambda v c^{\gamma})^n \frac{1}{\Gamma(n\gamma) n!} x^{n\gamma - 1}\right]\right\},
\end{aligned}$$

for all $x \in (0, \infty)$. Note that one can show that interchanging summation and differentiation in the second equality above is indeed allowed. However, the proof is omitted here.

(b) Let $X \sim f_X$ belong to the exponential dispersion family with $w, \phi, \theta, b(\cdot)$ and $c(\cdot, \cdot, \cdot)$ as given on the exercise sheet. Then, we have

$$\frac{x\theta}{\phi/w} \;=\; -xv \frac{(\gamma + 1)\left(\frac{\lambda v \gamma}{c}\right)^{-\frac{1}{\gamma+1}}}{\frac{\gamma+1}{\lambda\gamma}\left(\frac{\lambda v \gamma}{c}\right)^{\frac{\gamma}{\gamma+1}}} \;=\; -x\lambda v\gamma \left(\frac{\lambda v \gamma}{c}\right)^{-1} \;=\; -cx,$$

for all $x \ge 0$, and

$$\frac{b(\theta)}{\phi/w} \;=\; v \frac{\frac{\gamma+1}{\gamma}\left(\frac{-\theta}{\gamma+1}\right)^{-\gamma}}{\frac{\gamma+1}{\lambda\gamma}\left(\frac{\lambda v \gamma}{c}\right)^{\frac{\gamma}{\gamma+1}}} \;=\; \lambda v \frac{\left(\frac{\lambda v \gamma}{c}\right)^{\frac{\gamma}{\gamma+1}}}{\left(\frac{\lambda v \gamma}{c}\right)^{\frac{\gamma}{\gamma+1}}} \;=\; \lambda v.$$

Moreover, since

$$\frac{(\gamma+1)^{\gamma+1}}{\gamma}\left(\frac{\phi}{w}\right)^{-\gamma-1} = \frac{(\gamma+1)^{\gamma+1}}{\gamma}\left[\frac{\gamma+1}{\lambda v\gamma}\left(\frac{\lambda v\gamma}{c}\right)^{\frac{\gamma}{\gamma+1}}\right]^{-\gamma-1} = \frac{1}{\gamma}(\lambda v\gamma)^{\gamma+1}\left(\frac{\lambda v\gamma}{c}\right)^{-\gamma}$$
$$= \frac{1}{\gamma}\lambda v\gamma c^{\gamma} = \lambda v c^{\gamma},$$

we have, for all $x > 0$,

$$\begin{aligned} c(x,\phi,w) &= \log\left(\sum_{n=1}^{\infty}\left[\frac{(\gamma+1)^{\gamma+1}}{\gamma}\left(\frac{\phi}{w}\right)^{-\gamma-1}\right]^{n}\frac{1}{\Gamma(n\gamma)n!}\,x^{n\gamma-1}\right) \\ &= \log\left[\sum_{n=1}^{\infty}(\lambda v c^{\gamma})^{n}\frac{1}{\Gamma(n\gamma)n!}\,x^{n\gamma-1}\right]. \end{aligned}$$

By putting the above terms together, we get, for all $x > 0$,

$$\begin{aligned} f_X(x;\theta,\phi) &= \exp\left\{\frac{x\theta-b(\theta)}{\phi/w}+c(x,\phi,w)\right\} \\ &= \exp\left\{-(cx+\lambda v)+\log\left[\sum_{n=1}^{\infty}(\lambda v c^{\gamma})^{n}\frac{1}{\Gamma(n\gamma)n!}x^{n\gamma-1}\right]\right\} \\ &= f_S(x), \end{aligned}$$

and

$$f_X(0;\theta,\phi) = \exp\left\{\frac{0\cdot\theta-b(\theta)}{\phi/w}+c(0,\phi,w)\right\} = \exp\{-\lambda v\} = \mathbb{P}[S=0].$$

We conclude that S indeed belongs to the exponential dispersion family. Note that with this result at hand one might be tempted to estimate the shape parameter γ of the claim size distribution and to do a GLM analysis directly on the compound claim size S. However, there are two reasons to rather perform a separate GLM analysis of the claim frequency and the claim severity instead: First, claim frequency modelling is usually more stable than claim severity modelling and often much of the differences between tariff cells are due to the claim frequency. Second, a separate analysis of the claim frequency and the claim severity provides more insights into the differences between the tariffs.