ETH Zürich, D-MATH
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Non-Life Insurance: Mathematics and Statistics

Solution sheet 5

Solution 5.1 Large Claims

(a) The density of a Pareto distribution with threshold $\theta = 50$ and tail index $\alpha > 0$ is given by

$$f(x) = f_\alpha(x) = \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{-(\alpha+1)},$$

for all $x \geq \theta$. Using the independence of $Y_1, \ldots, Y_n$, the joint likelihood function $\mathcal{L}_\mathbf{Y}(\alpha)$ for the observation $\mathbf{Y} = (Y_1, \ldots, Y_n)$ can be written as

$$\mathcal{L}_\mathbf{Y}(\alpha) = \prod_{i=1}^{n} f_\alpha(Y_i) = \prod_{i=1}^{n} \frac{\alpha}{\theta}\left(\frac{Y_i}{\theta}\right)^{-(\alpha+1)} = \prod_{i=1}^{n} \alpha \theta^\alpha Y_i^{-(\alpha+1)},$$

whereas the joint log-likelihood function $\ell_\mathbf{Y}(\alpha)$ is given by

$$\ell_\mathbf{Y}(\alpha) = \log \mathcal{L}_\mathbf{Y}(\alpha) = \sum_{i=1}^{n} \log \alpha + \alpha \log \theta - (\alpha+1)\log Y_i = n \log \alpha + n\alpha \log \theta - (\alpha+1)\sum_{i=1}^{n} \log Y_i.$$

The MLE $\widehat{\alpha}_n^{\mathrm{MLE}}$ is defined as

$$\widehat{\alpha}_n^{\mathrm{MLE}} = \arg\max_{\alpha>0} \mathcal{L}_\mathbf{Y}(\alpha) = \arg\max_{\alpha>0} \ell_\mathbf{Y}(\alpha).$$

Calculating the first and the second derivative of $\ell_\mathbf{Y}(\alpha)$ with respect to α, we get

$$\frac{\partial}{\partial \alpha}\ell_\mathbf{Y}(\alpha) = \frac{n}{\alpha} + n\log\theta - \sum_{i=1}^{n}\log Y_i \quad \text{and}$$

$$\frac{\partial^2}{\partial \alpha^2}\ell_\mathbf{Y}(\alpha) = \frac{\partial}{\partial \alpha}\left(\frac{n}{\alpha} + n\log\theta - \sum_{i=1}^{n}\log Y_i\right) = -\frac{n}{\alpha^2} < 0,$$

for all $\alpha > 0$, from which we can conclude that $\ell_\mathbf{Y}(\alpha)$ is strictly concave in α. Thus, $\widehat{\alpha}_n^{\mathrm{MLE}}$ can be found by setting the first derivative of $\ell_\mathbf{Y}(\alpha)$ equal to 0. We get

$$\frac{n}{\widehat{\alpha}_n^{\mathrm{MLE}}} + n\log\theta - \sum_{i=1}^{n}\log Y_i = 0 \qquad \Longleftrightarrow \qquad \widehat{\alpha}_n^{\mathrm{MLE}} = \left(\frac{1}{n}\sum_{i=1}^{n}\log Y_i - \log\theta\right)^{-1}.$$

(b) Let $\widehat{\alpha}$ denote the unbiased version of the MLE for the storm and flood data given in Table 1 of the exercise sheet. Since we observed 15 storm and flood events, we have $n = 15$. Thus, $\widehat{\alpha}$ can be calculated as

$$\widehat{\alpha} = \frac{n-1}{n}\left(\frac{1}{n}\sum_{i=1}^{n}\log Y_i - \log\theta\right)^{-1} = \frac{14}{15}\left(\frac{1}{15}\sum_{i=1}^{15}\log Y_i - \log 50\right)^{-1} \approx 0.98,$$

where for $Y_1, \ldots, Y_{15}$ we plugged in the observed claim sizes given in Table 1 of the exercise sheet. Note that with $\widehat{\alpha} = 0.98 < 1$ the expectation of the claim sizes does not exist.

(c) We define $N_1, \ldots, N_{20}$ to be the numbers of yearly storm and flood events for the twenty years $1986 - 2005$. By assumption, $N_1, \ldots, N_{20}$ are i.i.d. Poisson distributed with frequency parameter λ. Using Estimator 2.32 of the lecture notes (version of December 17, 2020) with $v_1 = \cdots = v_{20} = 1$, the MLE $\widehat{\lambda}$ of λ is given by

$$\widehat{\lambda} \;=\; \frac{1}{\sum_{i=1}^{20} 1} \sum_{i=1}^{20} N_i \;=\; \frac{1}{20} \sum_{i=1}^{20} N_i.$$

Since we observed 15 storm and flood events in total, we get

$$\widehat{\lambda} \;=\; \frac{15}{20} \;=\; 0.75.$$

(d) Using Proposition 2.11 of the lecture notes (version of December 17, 2020), the expected yearly claim amount $\mathbb{E}[S]$ of storm and flood events with maximal claims cover M is given by

$$\mathbb{E}[S] \;=\; \lambda \, \mathbb{E}[\min\{Y_1, M\}].$$

The expected value of $\min\{Y_1, M\}$ can be calculated as

$$
\begin{aligned}
\mathbb{E}[\min\{Y_1, M\}] &= \mathbb{E}[\min\{Y_1, M\} 1_{\{Y_1 \le M\}}] + \mathbb{E}[\min\{Y_1, M\} 1_{\{Y_1 > M\}}] \\
&= \mathbb{E}[Y_1 1_{\{Y_1 \le M\}}] + \mathbb{E}[M 1_{\{Y_1 > M\}}] \\
&= \mathbb{E}[Y_1 1_{\{Y_1 \le M\}}] + M \mathbb{P}[Y_1 > M],
\end{aligned}
$$

where for $\mathbb{E}[Y_1 1_{\{Y_1 \le M\}}]$ and $M \mathbb{P}[Y_1 > M]$ we have

$$
\begin{aligned}
\mathbb{E}[Y_1 1_{\{Y_1 \le M\}}] &= \int_\theta^\infty x 1_{\{x \le M\}} f(x)\, dx = \int_\theta^M x \frac{\alpha}{\theta} \left(\frac{x}{\theta}\right)^{-(\alpha+1)} dx = \alpha \theta^\alpha \left[\frac{1}{1-\alpha} x^{1-\alpha}\right]_\theta^M \\
&= \frac{\alpha}{1-\alpha} \theta^\alpha M^{1-\alpha} - \frac{\alpha}{1-\alpha} \theta = \frac{\alpha}{1-\alpha} \theta \left(\frac{M}{\theta}\right)^{1-\alpha} - \frac{\alpha}{1-\alpha} \theta \\
&= \theta \frac{\alpha}{1-\alpha} \left[\left(\frac{M}{\theta}\right)^{1-\alpha} - 1\right]
\end{aligned}
$$

and

$$
M \mathbb{P}[Y_1 > M] \;=\; M(1 - \mathbb{P}[Y_1 \le M]) \;=\; M\left(1 - \left[1 - \left(\frac{M}{\theta}\right)^{-\alpha}\right]\right) \;=\; \theta \left(\frac{M}{\theta}\right)^{1-\alpha}.
$$

Hence, we get

$$
\mathbb{E}[\min\{Y_1, M\}] \;=\; \theta \frac{\alpha}{1-\alpha} \left[\left(\frac{M}{\theta}\right)^{1-\alpha} - 1\right] + \theta \left(\frac{M}{\theta}\right)^{1-\alpha} \;=\; \theta \frac{1}{1-\alpha} \left(\frac{M}{\theta}\right)^{1-\alpha} - \theta \frac{\alpha}{1-\alpha}.
$$

Replacing the unknown parameters by their estimates, we get for the estimated expected total yearly claim amount $\widehat{\mathbb{E}}[S]$:

$$
\widehat{\mathbb{E}}[S] = \widehat{\lambda} \left[\frac{\theta}{1-\widehat{\alpha}} \left(\frac{M}{\theta}\right)^{1-\widehat{\alpha}} - \frac{\theta \cdot \widehat{\alpha}}{1-\widehat{\alpha}}\right] \approx 0.75 \left[\frac{50}{1-0.98} \left(\frac{2'000}{50}\right)^{1-0.98} - \frac{50 \cdot 0.98}{1-0.98}\right] \approx 180.4.
$$

(e) According to our compound Poisson model, next year's total yearly claim amount $S \sim \mathrm{CompPoi}(\lambda, G)$ of storm and flood events with claim amounts exceeding CHF 50 million can be written as

$$S \;=\; \sum_{i=1}^{N} Y_i,$$

where $N \sim \text{Poi}(\lambda)$, $Y_1, Y_2, \ldots$ are i.i.d. with distribution function G, and N and $Y_1, Y_2, \ldots$ are independent. Since we are only interested in events that exceed the level of CHF $M = 2$ billion, we define S_M as

$$S_M = \sum_{i=1}^{N} Y_i 1_{\{Y_i > M\}}.$$

Due to Theorem 2.14 of the lecture notes (version of December 17, 2020), we have $S_M \sim \text{CompPoi}(\lambda_M, G_M)$ for some distribution function G_M and

$$\lambda_M = \lambda \mathbb{P}[Y_1 > M] = \lambda(1 - \mathbb{P}[Y_1 \leq M]) = \lambda \left(1 - \left[1 - \left(\frac{M}{\theta}\right)^{-\alpha}\right]\right) = \lambda \left(\frac{M}{\theta}\right)^{-\alpha}.$$

Defining a random variable $N_M \sim \text{Poi}(\lambda_M)$, the probability that we observe at least one storm and flood event next year which exceeds the level of CHF $M = 2$ billion is given by

$$\mathbb{P}[N_M \geq 1] = 1 - \mathbb{P}[N_M = 0] = 1 - \exp\{-\lambda_M\} = 1 - \exp\left\{-\lambda \left(\frac{M}{\theta}\right)^{-\alpha}\right\}.$$

If we replace the unknown parameters by their estimates, this probability can be estimated by

$$\widehat{\mathbb{P}}[N_M \geq 1] = 1 - \exp\left\{-\widehat{\lambda} \left(\frac{M}{\theta}\right)^{-\hat{\alpha}}\right\} \approx 1 - \exp\left\{-0.75 \left(\frac{2'000}{50}\right)^{-0.98}\right\} \approx 0.02.$$

Note that, in particular, such a storm and flood event that exceeds the level of CHF 2 billion is expected roughly every $1/0.02 = 50$ years.

Solution 5.2 Claim Size Distributions

The R code used to generate the four i.i.d. samples is given in Listing 1.

Listing 1: R code for Exercise 5.2 (Data generation).

```
1   ### Size of the i.i.d. samples
2   n <- 10000
3
4   ### Generate the gamma i.i.d. sample
5   gamma <- 1/4
6   c <- 1/40000
7   set.seed(100)
8   gamma.sample <- rgamma(n=n, shape=gamma, rate=c)
9
10  ### Generate the Weibull i.i.d. sample
11  tau <- 0.54
12  c <- 0.000175
13  set.seed(200)
14  weibull.sample <- rgamma(n=n, shape=1, rate=1)^(1/tau)/c
15
16  ### Generate the log-normal i.i.d. sample
17  mu <- log(2000*sqrt(5))
18  sigma.squared <- log(5)
19  set.seed(300)
20  lognormal.sample <- exp(rnorm(n=n, mean=mu, sd=sqrt(sigma.squared)))
21
22  ### Generate the Pareto i.i.d. sample
23  theta <- 10000*(sqrt(5)/(2+sqrt(5)))
24  alpha <- 1+sqrt(5)/2
25  set.seed(400)
26  pareto.sample <- theta*exp(rgamma(n=n, shape=1, rate=alpha))
```

In Figure 1 (generated by the R code given in Listing 2) we show the densities (left) of the generated i.i.d. samples as well as the corresponding box plots (right), both on a log scale. We only display

logarithmic values starting from 0. We see for example that we have a lot of very small values in case of the gamma distribution and the Weibull distribution. The smallest values observed are considerably bigger for the log-normal and especially the Pareto distribution. Moreover, the value of the biggest value observed increases in going from the gamma over the Weibull and the log-normal to the Pareto distribution. We cannot say much about the tails from looking at these two plots.
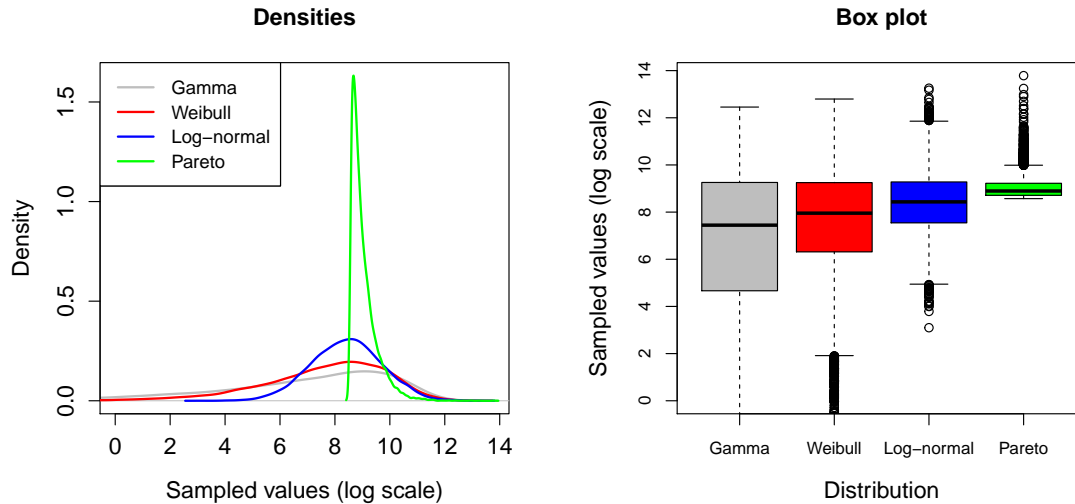

Figure 1: Plot of the densities of the four i.i.d. samples (left). Box plots of the four i.i.d. samples (right).

Listing 2: R code for Exercise 5.2 (Figure 1).

```
1   ### Densities
2   ymax <- max(density(log(gamma.sample))$y, density(log(weibull.sample))$y,
3                density(log(lognormal.sample))$y, density(log(pareto.sample))$y)
4   ymax2 <- max(log(gamma.sample),log(weibull.sample),log(lognormal.sample),log(pareto.sample))
5   plot(density(log(gamma.sample)), xlim=c(0,ymax2), col="grey", ylim=c(0,ymax), main="Densities",
6                xlab="Sampled values (log scale)", cex.lab=1.25, cex.main=1.25, cex.axis=1.25,
7                lwd=2)
8   lines(density(log(weibull.sample)), col="red", xlim=c(0,ymax2), lwd=2)
9   lines(density(log(lognormal.sample)), col="blue", xlim=c(0,ymax2), lwd=2)
10  lines(density(log(pareto.sample)), col="green", xlim=c(0,ymax2), lwd=2)
11  legend("topleft", lty=1, lwd=2, col=c("grey","red","blue","green"),
12         legend=c("Gamma","Weibull","Log-normal","Pareto"))
13
14  ### Box plots
15  boxplot(log(gamma.sample), log(weibull.sample), log(lognormal.sample), log(pareto.sample),
16          ylim=c(0,ymax2), col=c("grey","red","blue","green"), main="Box plot",
17          names=c("Gamma","Weibull","Log-normal","Pareto"), xlab="Distribution",
18          ylab="Sampled values (log scale)", cex.lab=1.25, cex.main=1.25, cex.axis=0.95)
```

In Figure 2 (generated by the R code given in Listing 3) we show the plots of the empirical distribution functions (left, on a log scale) and of the empirical loss size index functions (right) of the generated i.i.d. samples. For the plot of the empirical distribution functions we only display logarithmic values starting from 0. We observe that the empirical distribution functions almost perfectly intersect at the point with x-coordinate equal to log(10'000)≈ 9.21. This means that for all of the four considered distributions approximately the same percentage of observations is smaller than the expected value. This percentage is roughly equal to 75%, indicating that three quarters of the observations are smaller than the expected value and one quarter of the observations are above the expected value. Thus, not surprisingly, the large claims are the main driver of the expected value. We get confirmed the observations from Figure 1, namely that the smallest values observed

are considerably bigger for the log-normal and especially the Pareto distribution, compared to the gamma and the Weibull distribution. This carries over to the plot of the empirical loss size index function. Also these two plots do not tell us much about the tails of the distributions.
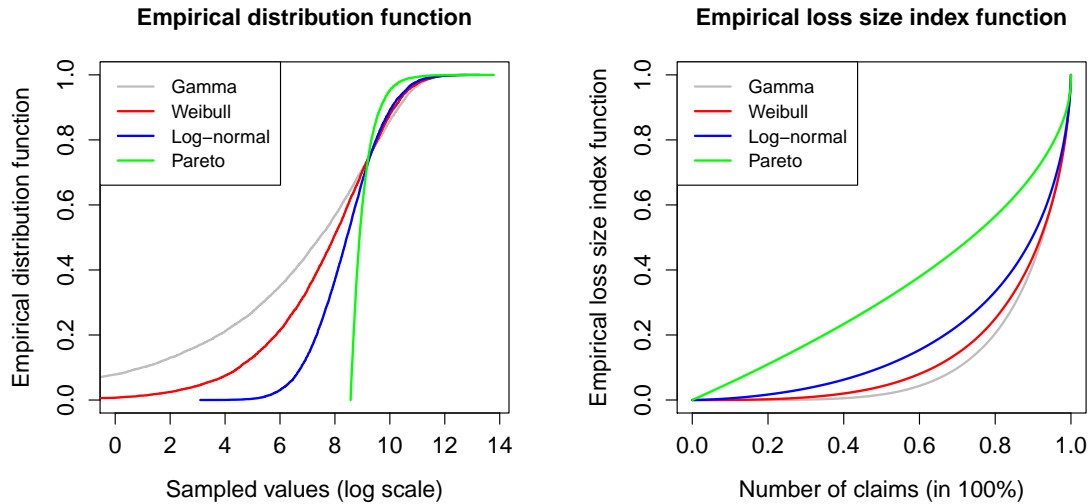

Figure 2: Plot of the empirical distribution functions of the four i.i.d. samples (left). Plot of the empirical loss size index functions of the four i.i.d. samples (right).

Listing 3: R code for Exercise 5.2 (Figure 2).

```
1  ### Empirical distribution functions
2  plot(log(gamma.sample[order(gamma.sample)]), 1:n/(n+1), xlim=c(0,ymax2), type="l", col="grey",
3      main="Empirical distribution function", xlab="Sampled values (log scale)",
4      ylab="Empirical distribution function", cex.lab=1.25, cex.main=1.25, cex.axis=1.25, lwd=2)
5  lines(log(weibull.sample[order(weibull.sample)]), 1:n/(n+1), xlim=c(0,ymax2), col="red", lwd=2)
6  lines(log(lognormal.sample[order(lognormal.sample)]), 1:n/(n+1), xlim=c(0,ymax2), col="blue",
7      lwd=2)
8  lines(log(pareto.sample[order(pareto.sample)]), 1:n/(n+1), xlim=c(0,ymax2), col="green", lwd=2)
9  legend("topleft", lty=1, lwd=2, col=c("grey","red","blue","green"),
10      legend=c("Gamma","Weibull","Log-normal","Pareto"))
11
12 ### Empirical loss size index functions
13 plot(1:n/n, cumsum(gamma.sample[order(gamma.sample)])/sum(gamma.sample), type="l", col="grey",
14      main="Empirical loss size index function", xlab="Number of claims (in 100%)",
15      ylab="Empirical loss size index function", cex.lab=1.25, cex.main=1.25, cex.axis=1.25,
16      lwd=2)
17 lines(1:n/n, cumsum(weibull.sample[order(weibull.sample)])/sum(weibull.sample), type="l",
18      col="red", lwd=2)
19 lines(1:n/n, cumsum(lognormal.sample[order(lognormal.sample)])/sum(lognormal.sample), type="l",
20      col="blue", lwd=2)
21 lines(1:n/n, cumsum(pareto.sample[order(pareto.sample)])/sum(pareto.sample), type="l",
22      col="green", lwd=2)
23 legend("topleft", lty=1, lwd=2, col=c("grey","red","blue","green"),
24      legend=c("Gamma","Weibull","Log-normal","Pareto"))
```

In Figure 3 (generated by the R code given in Listing 4) we show the log-log plots (left) and the plot of the empirical mean excess functions (right) of the generated i.i.d. samples. These two plots can be used for studying the tails of the distributions. We see in both plots that the gamma distribution is the most light-tailed distribution. The Weibull distribution and the log-normal distribution have a similar tail behaviour, with slightly heavier tails of the log-normal distribution. Note that this similar tail behaviour is due to the value of the parameter τ of the Weibull distribution being smaller than 1. With a value $\tau \geq 1$ the distribution gets (even) more light-tailed. The most heavy-tailed distribution among the four distributions we analyzed here is the Pareto distribution.

Listing 4: R code for Exercise 5.2 (Figure 3).

```
1  ### Log-log plots
2  plot(log(gamma.sample[order(gamma.sample)]), log(1-1:n/(n+1)), xlim=c(0,ymax2), type="l",
3       col="grey", main="Log-log plot", xlab="log(sampled values)",
4       ylab="log(1-empirical distribution function)", cex.lab=1.25, cex.main=1.25, cex.axis=1.25,
5       lwd=2)
6  lines(log(weibull.sample[order(weibull.sample)]), log(1-1:n/(n+1)), xlim=c(0,ymax2), type="l",
7       col="red", lwd=2)
8  lines(log(lognormal.sample[order(lognormal.sample)]), log(1-1:n/(n+1)), xlim=c(0,ymax2),
9       type="l", col="blue", lwd=2)
10 lines(log(pareto.sample[order(pareto.sample)]), log(1-1:n/(n+1)), xlim=c(0,ymax2), type="l",
11      col="green", lwd=2)
12 legend("bottomleft", lty=1, lwd=2, col=c("grey","red","blue","green"),
13      legend=c("Gamma","Weibull","Log-normal","Pareto"))
14
15 ### Empirical mean excess functions
16 mean.excess.function <- Vectorize(function(threshold,input.sample){
17 mean(input.sample[input.sample>threshold])-threshold
18 },"threshold")
19 xmax <- pareto.sample[order(pareto.sample)][n-1]
20 ymax3 <- max(pareto.sample)-xmax
21 plot(gamma.sample[order(gamma.sample)][-n],
22      mean.excess.function(gamma.sample[order(gamma.sample)][-n],gamma.sample), pch=16,
23      col="grey", xlim=c(0,xmax), ylim=c(0,ymax3), main="Empirical mean excess function",
24      xlab="Threshold", ylab="Mean excess function", cex.lab=1.25, cex.main=1.25, cex.axis=1.25)
25 points(weibull.sample[order(weibull.sample)][-n],
26      mean.excess.function(weibull.sample[order(weibull.sample)][-n],weibull.sample), pch=16,
27      col="red", ylim=c(0,ymax3))
28 points(lognormal.sample[order(lognormal.sample)][-n],
29      mean.excess.function(lognormal.sample[order(lognormal.sample)][-n],lognormal.sample),
30      pch=16, col="blue", ylim=c(0,ymax3))
31 points(pareto.sample[order(pareto.sample)][-n],
32      mean.excess.function(pareto.sample[order(pareto.sample)][-n],pareto.sample),pch=16,
33      col="green", ylim=c(0,ymax3))
34 legend("topleft", pch=16, col=c("grey","red","blue","green"),
35      legend=c("Gamma","Weibull","Log-normal","Pareto"))
```
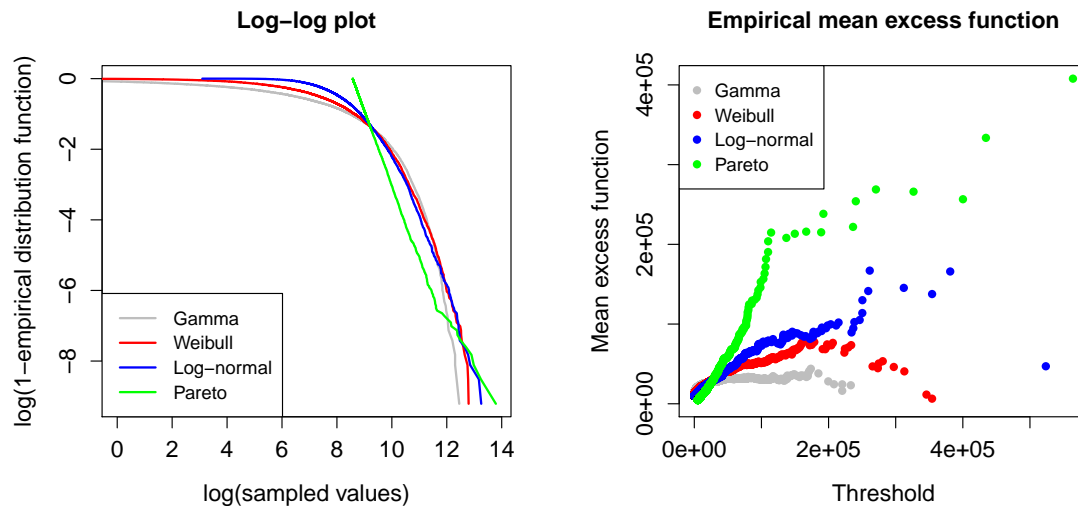

Figure 3: Log-log plots of the four i.i.d. samples (left). Plot of the empirical mean excess functions of the four i.i.d. samples (right).

Summarizing, we can say that although we fixed the mean and the standard deviation to be the same, all of the four considered distributions behave differently. This implies that one has to carefully select the appropriate claim size distribution.

Solution 5.3 Hill Estimator

The Hill plot (on the left, generated by the R code of Listing 5) and the log-log plot (on the right, generated by the R code of Listing 6) are given in Figure 4. Even though we sampled from a Pareto distribution with tail index $\alpha = 2$, it is not at all clear to see that the data comes from a Pareto distribution. In the Hill plot we see that, first, the estimates of α seem more or less correct, but starting from the 180 largest observations, the plot suggests a higher α or even another distribution. In the log-log plot we see that for small-sized and medium-sized claims the fit seems to be fine. But looking at the largest claims, we would conclude that our data is not as heavy-tailed as a true Pareto distribution with threshold $\theta = 10$ and tail index $\alpha = 2$ would suggest. We observe these problems even though we sampled directly from a Pareto distribution. This indicates the difficulties one faces when trying to fit such a distribution to a real data set, where we often have less observations than in this example and the observations may be contaminated by other distributions.

Listing 5: R code for Exercise 5.3 (Hill plot).

```
1  hill.plot.function <- function(n, theta, alpha, seed1){
2    set.seed(seed1)
3    data.1 <- rgamma(n, shape=1, scale=1/alpha)
4    data <- theta * exp(data.1)
5    log.data.ordered <- log(data[order(data, decreasing=FALSE)])
6    n.obs <- n:5
7    hill.estimator <- ((sum(log.data.ordered)-cumsum(log.data.ordered)
8                        +log.data.ordered)[-((n-3):n)]/n.obs-log.data.ordered[-((n-3):n)])^(-1)
9    upper.bound <- hill.estimator+sqrt(n.obs^2/((n.obs-1)^2*(n.obs-2))*hill.estimator^2)
10   lower.bound <- hill.estimator-sqrt(n.obs^2/((n.obs-1)^2*(n.obs-2))*hill.estimator^2)
11   plot(hill.estimator, ylim=c(min(hill.estimator)-1,max(hill.estimator)+1), xaxt="n",
12        xlab="Number of observations", ylab="Pareto tail index parameter",
13        main="Hill plot for alpha", cex=0.5, cex.lab=1.25, cex.main=1.25, cex.axis=1.25)
14   axis(1, at=c(1,seq(from=n/10+1, to=n*9/10+1, by=n/10), n-5), c(seq(from=n, to=n/10,
15        by=-n/10), 5))
16   lines(upper.bound)
17   lines(lower.bound)
18   abline(h=alpha, col="blue", lwd=2)
19   legend("topleft", col=c("blue","black"), lty=c(1,NA), pch=c(NA,1), lwd=c(2,NA),
20          legend=c("true tail index","estimated tail index"))
21 }
22
23 hill.plot.function(n=300,theta=10,alpha=2,seed1=100)
```
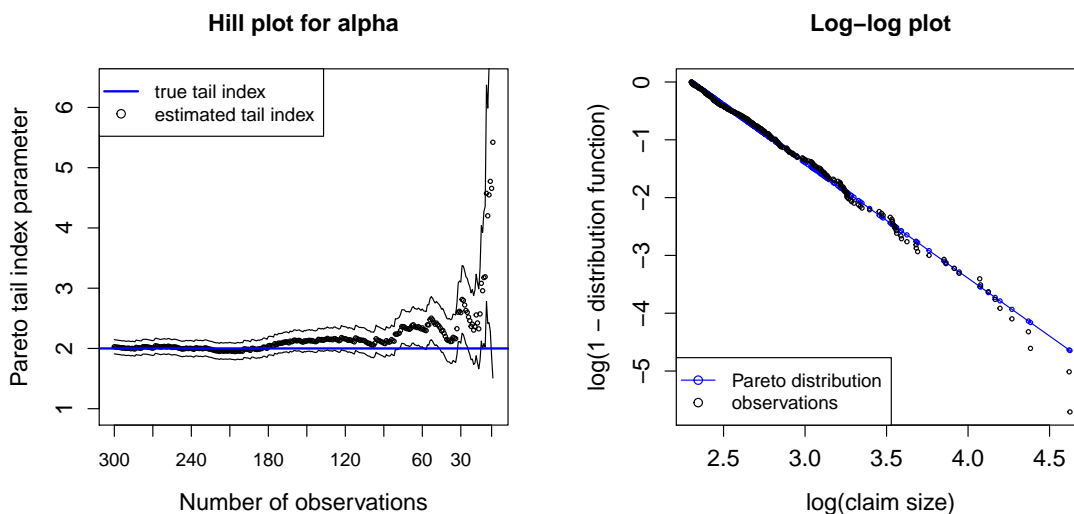

Figure 4: Hill plot for determining the tail index α (left). Log-log plot for the observations and the Pareto distribution (right).

Listing 6: R code for Exercise 5.3 (Log-log plot).

```
1   log.log.plot.function <- function(n, theta, alpha, seed1){
2     set.seed(seed1)
3     data.1 <- rgamma(n, shape=1, scale=1/alpha)
4     data <- theta*exp(data.1)
5     data.ordered <- data[order(data, decreasing=FALSE)]
6     log.data.ordered <- log(data.ordered)
7     true.sf <- (data.ordered/theta)^(-alpha)
8     empirical.sf <- 1-(1:n)/(n+1)
9     plot(log.data.ordered, log(true.sf), xlab="log(claim size)",
10          ylab="log(1 - distribution function)",
11          ylim=c(min(log(true.sf),log(empirical.sf)),max(log(true.sf),log(empirical.sf))),
12          main="Log-log plot", cex.lab=1.25, cex.main=1.25, cex.axis=1.25, cex=0.5, col="blue")
13     lines(log.data.ordered,log(true.sf), col="blue")
14     points(log.data.ordered, log(empirical.sf), col="black", cex=0.5)
15     legend("bottomleft", col=c("blue","black"), lty=c(1,NA), pch=c(1,1),
16            legend=c("Pareto distribution","observations"))
17   }
18
19   log.log.plot.function(n=300,theta=10,alpha=2,seed1=100)
```

Solution 5.4 Pareto Distribution

The density g and the distribution function G of $Y \sim \mathrm{Pareto}(\theta, \alpha)$ are defined by

$$g(x) = \frac{\alpha}{\theta} \left(\frac{x}{\theta}\right)^{-(\alpha+1)} \qquad \text{and} \qquad G(x) = 1 - \left(\frac{x}{\theta}\right)^{-\alpha},$$

respectively, for all $x \geq \theta$.

(a) The survival function $\bar{G} = 1 - G$ of Y is given by

$$\bar{G}(x) = 1 - G(x) = \left(\frac{x}{\theta}\right)^{-\alpha},$$

for all $x \geq \theta$. Hence, for all $t > 0$ we have

$$\lim_{x \to \infty} \frac{\bar{G}(xt)}{\bar{G}(x)} = \lim_{x \to \infty} \frac{(xt/\theta)^{-\alpha}}{(x/\theta)^{-\alpha}} = t^{-\alpha}.$$

Thus, by definition, the survival function of Y is regularly varying at infinity with tail index α.

(b) Let $\theta \leq u_1 < u_2$. Then, the expected value of Y within the layer $(u_1, u_2]$ can be calculated as

$$\mathbb{E}[Y 1_{\{u_1 < Y \leq u_2\}}] = \int_{\theta}^{\infty} x 1_{\{u_1 < x \leq u_2\}} g(x)\, dx = \int_{u_1}^{u_2} x \frac{\alpha}{\theta} \left(\frac{x}{\theta}\right)^{-(\alpha+1)} dx$$

$$= \alpha \theta \int_{u_1}^{u_2} \frac{1}{\theta} \left(\frac{x}{\theta}\right)^{-\alpha} dx.$$

In the case $\alpha \neq 1$, we get

$$\mathbb{E}[Y 1_{\{u_1 < Y \leq u_2\}}] = \alpha \theta \left[-\frac{1}{\alpha - 1} \left(\frac{x}{\theta}\right)^{-\alpha+1}\right]_{u_1}^{u_2} = \theta \frac{\alpha}{\alpha - 1} \left[\left(\frac{u_1}{\theta}\right)^{-\alpha+1} - \left(\frac{u_2}{\theta}\right)^{-\alpha+1}\right],$$

and if $\alpha = 1$, we get

$$\mathbb{E}[Y 1_{\{u_1 < Y \leq u_2\}}] = \theta \int_{u_1}^{u_2} \frac{1}{x} dx = \theta \log\left(\frac{u_2}{u_1}\right).$$

(c) Let $\alpha > 1$ and $y > \theta$. Then, the expected value μ_Y of Y is given by

$$\mu_Y = \theta \frac{\alpha}{\alpha - 1}$$

and, similarly as in part (b), we get

$$\mathbb{E}[Y 1_{\{Y \leq y\}}] = \mathbb{E}[Y 1_{\{\theta < Y \leq y\}}] = \theta \frac{\alpha}{\alpha - 1} \left[\left(\frac{\theta}{\theta} \right)^{-\alpha+1} - \left(\frac{y}{\theta} \right)^{-\alpha+1} \right] = \mu_Y \left[1 - \left(\frac{y}{\theta} \right)^{-\alpha+1} \right].$$

Hence, for the loss size index function for level $y > \theta$ we have

$$\mathcal{I}[G(y)] = \frac{1}{\mu_Y} \mathbb{E}[Y 1_{\{Y \leq y\}}] = 1 - \left(\frac{y}{\theta} \right)^{-\alpha+1} \in (0, 1).$$

(d) Let $\alpha > 1$ and $u > \theta$. The mean excess function of Y above u can be calculated as

$$e(u) = \mathbb{E}[Y - u | Y > u] = \mathbb{E}[Y | Y > u] - u = \frac{\mathbb{E}[Y 1_{\{Y > u\}}]}{\mathbb{P}[Y > u]} - u = \frac{\mathbb{E}[Y 1_{\{Y > u\}}]}{\bar{G}(u)} - u,$$

where for $\mathbb{E}[Y 1_{\{Y > u\}}]$ we have, similarly as in part (b),

$$\mathbb{E}[Y 1_{\{Y > u\}}] = \mathbb{E}[Y 1_{\{u < Y \leq \infty\}}] = \alpha \theta \left[-\frac{1}{\alpha - 1} \left(\frac{x}{\theta} \right)^{-\alpha+1} \right]_u^\infty = \frac{\alpha}{\alpha - 1} \theta \left(\frac{u}{\theta} \right)^{-\alpha+1}$$

$$= \frac{\alpha}{\alpha - 1} u \bar{G}(u).$$

Thus, we get

$$e(u) = \frac{\alpha}{\alpha - 1} u - u = \frac{1}{\alpha - 1} u.$$

Note that the mean excess function $u \mapsto e(u)$ has slope $\frac{1}{\alpha-1} > 0$.