Valentin Bosshard
Patricia Dietzsch



# Overview

*Abstract:* In this seminar we will define quasimorphisms and use them as an algebraic tool to study various automorphism groups of manifolds. After a short introduction to symplectic geometry, we will mainly focus on the group of Hamiltonian diffeomorphisms and the Calabi quasimorphism.

*Objective:* By giving two half-hour talks, typing short summaries for those talks and participating in talks by others, each participant will get familiar with the concept of quasimorphisms, learn about some concrete examples in and outside the world of symplectic geometry, as well as develop presentation and collaboration skills.

*Organisation:* Each student gives two talks, one in the first half, and one in the second half of the semester. On Fridays, the student giving the talk on the following Wednesday should discuss her/his first draft of the talk with Patricia or Valentin. A short summary of the main definitions and statements should be typed into the common overleaf project by the day of the talk.

*Prerequisites/Notes:* Knowledge of Differential Geometry I and Algebraic Topology I.

## Talk 0 - 22.9.:

***Topic:*** **Automorphism Groups and Quasimorphisms.**
*Speakers:* Valentin Bosshard, Patricia Dietzsch
*Content:*

- Definition of various automorphism groups.
- Some known theorems about automorphism groups with focus on simplicity and perfectness.
- Quasimorphisms: what they are and why they are useful to study automorphism groups.

## Talk 1 - 6.10. (*easy*):

***Topic:*** **Introduction to Quasimorphisms.**
*References:* [Car13, Chapter 3.1], [PR14, Chapter 3.1], [GG04, Introduction]
*Content:*

- Definition of quasimorphisms, defect, homogeneous quasimorphisms.
- Explain the bijection between the space of quasimorphisms modulo bounded functions and homogeneous quasimorphisms (see e.g. [Car13, Proposition 3.1.4]).
- Definition of simple and perfect groups, commutator and commutator length.
- Show that if a non-trivial homogeneous quasimorphism exists then the commutator length is unbounded (see e.g. [GG04, Introduction]).
- Give a few examples of automorphism groups which are simple/not simple, perfect/not perfect.

*Mentor:* Patricia

Talk 2 - 13.10. (*medium*):

***Topic:*** **The Homeomorphism group of the circle.**
*References:* [Man15, Chapter 1.3], [Ghy01, Chapter 4]
*Content:*

- Definition of $\mathrm{Homeo}_0(S^1)$.
- Why is $\mathrm{Homeo}_0(S^1) = \mathrm{Homeo}_+(S^1)$, where the latter denotes the group of orientation-preserving homeomorphisms? (Look at graph of the maps)
- Indicate the general strategy for proving simplicity and perfectness of automorphism groups.
- Explain the fragmentation property.
- A connected topological group is generated by a neighborhood of the identity.
- $\mathrm{Homeo}_0(S^1)$ has the framentation property for intervals [Ghy01, hidden in the end of proof of Theorem 4.3].
- The fragmentation property of a homeomorphism group implies perfect and simple [Man15, Corollary 1.9,1.10].
- Conclude that $\mathrm{Homeo}_0(S^1)$ is perfect and simple.

*Mentor:* Valentin

Talk 3 - 20.10. (*easy*):

***Topic:*** **Rotation and translation numbers.**
*References:* [Ghy01, Chapter 5]
*Content:*

- Definition of universal cover $\widetilde{\mathrm{Homeo}_0}(S^1)$.
- Definition of translation number and proof that it is a quasimorphism.
- It is the only quasimorphism (Proposition 5.4).
- Definition of rotation number.
- If time allows: Rotation number is rational iff there is a periodic point [Mer19, Proposition 16.1].

*Mentor:* Valentin

Talk 4 - 27.10. (*medium*):

***Topic:*** **Quasimorphisms on the automorphism group of the disc.**
*References:* [GG04, sections 2.1, 5.1 and 5.2]
*Content:*

- Definition of $\mathrm{Diff}_0(D^2, \partial D^2, \mathrm{area})$.
- Explain the construction of Ruelle's invariant on $\mathrm{Diff}_0(D^2, \partial D^2, \mathrm{area})$ (section 2.1).
- Show that it is a non-trivial quasimorphism.
- Define the pure braid group $P_n(D^2)$ (section 5.1).
- Explain how a quasimorphism on $P_n(D^2)$ gives rise to a quasimorphism on $\mathrm{Diff}_0(D^2, \partial D^2, \mathrm{area})$ (section 5.2, see also [BKS18, pages 2 and 3]).

*Mentor:* Patricia

Talk 5 - 3.11. (*medium*):

***Topic:*** **Quasimorphisms on the automorphism groups of closed oriented surfaces.**
*References:* [GG04]
*Content:*

- State the existence result [GG04, Theorem 1.2].
- Explain the construction of Ruelle's invariant on $\mathrm{Diff}_0(T^2, \mathrm{area})$ (section 2.2).
- Summarize the idea for constructing a quasimorphism on $\mathrm{Diff}_0(\Sigma_g, \mathrm{area})$ for $g \geq 2$ (section 2.3).

*Mentor:* Patricia

Talk 6 - 10.11. (*easy*):

***Topic:* Introduction to symplectic geometry and the group of Hamiltonian diffeomorphisms.**
*References:* [MS17, Chapter 3], ([Pol12, Chapter 1], [Car13, Chapter 2.1]) .
*Content:*

- Definition of symplectic manifolds. Examples $\mathbb{R}^{2n}$ and surfaces.
- Statement of Darboux theorem without proof.
- Symplectic manifolds are orientable, even-dimensional.
- Definition of symplectomorphisms and $\mathrm{Symp}(M)$.
- Symplectic vs Hamiltonian vector fields (repeat Cartan's formula for the Lie derivative).
- Definition of Hamiltonian diffeomorphisms and $\mathrm{Ham}(M)$.
- Example of height function on sphere ([MS17, Example 3.1.7]).
- Proof that $\mathrm{Ham}(M)$ is a normal subgroup of $\mathrm{Symp}_0(M)$ ([MS17, Exercise 3.1.14]). Maybe subsection *Hamiltonian symplectomorphisms* until Exercise 10.1.3 also helps.

*Mentor:* Valentin

Talk 7 - 17.11. (*easy*):

***Topic:* The Flux homomorphism.**
*References:* [MS17, Chapter 10.2], [Car13, Chapter 2.2].
*Content:*

- Definition of exact symplectic manifolds.
- Characterization of Hamiltonian isotopies [MS17, Proposition 9.3.1, Corollary 9.3.3] or [Car13, Lemma 2.2.1].
- Definition of the flux homomorphism $\widetilde{\mathrm{Symp}}_0(M, \omega) \to H^1(M)$.
- (Compare to the flux defined in talk 4 and 5 in the two-dimensional case).
- Flux characterizes Hamiltonian isotopies ([MS17, Theorem 10.2.5]).
- Exact sequences induced by flux ([MS17, Proposition 10.2.13]).

*Mentor:* Valentin

Talk 8 - 24.11. (*easy*):

***Topic:* The Calabi homomorphism and Calabi quasimorphisms.**
*References:* [MS17, Chapter 10.3], [Car13, Chapter 2.2, 3.2].
*Content:*

- Definition Calabi homomorphism on $\mathrm{Ham}_c(M)$ for exact symplectic manifolds $M$.
- Well-definedness, homomorphism property and uniqueness.
- (Calabi for non-compact, non-exact symplectic manifolds.)
- $\mathrm{Ham}_c(M)$ is not simple for $(M, \omega)$ exact, but the kernel of the Calabi homomorphism is simple ([Ban97] without proof).
- $\mathrm{Ham}(M)$ is simple ([Ban97] without proof). This motivates the definition of Calabi quasimorphisms.
- Explain why the quasimorphisms from talks 4 and 5 are non-trivial on the kernel of the Calabi homomorphism resp. the flux homomorphism ([GG04], careful: the flux homomorphism is also called Calabi homomorphism in [GG04]).
- List possible requirements on the definition Calabi quasimorphisms that one might wish to have ([Py06b, page 178 starting at question in italic up to page 179] which refers to [EP03]).

*Mentor:* Valentin

Talk 9 - 1.12. (*medium*):

***Topic:* Calabi quasimorphisms on $\mathrm{Ham}(T^2)$.**
*References:* [Py06a] (in French!)
*Content:*

- Construct the homogeneous quasi-morphism $C_\phi$ (see second half of section 2 in [Py06a] and Talk 4)
- State Theorem 2.1 from [Py06a] and explain how this provides a construction for Calabi quasimorphisms.
- Prove it (see [Py06a, section 2]).
- Define $\mathcal{F}$ and the Reeb graph.
- State Theorem 0.2
- Outline its proof ([Py06a, section 3])).

*Mentor: Patricia and Valentin*

Talk 10 - 8.12. (*hard*):

***Topic:*** **Calabi quasimorphisms on $\mathrm{Ham}(\Sigma_g)$, $g \geq 2$.**
*References:* [Py06b] (in French!).
*Content:*

- State Théorème 1 in [Py06b] on the existence of Calabi quasimorphisms.
- Explain the construction of the Calabi quasimorphism $\mathrm{Cal}_S$ on page 183.
- Explain why this quasimorphism is equal to the Calabi homomorphism for diffeomorphisms supported in a disc (not the annulus) (pages 184-2nd formula on page 185).
- Define $\mathcal{F}$ and the Reeb graph (page 179).
- State Theorem 2 giving an explicit formula for autonomous Hamiltonian diffeomorphisms.
- Outline its proof ([Py06b, section 2.2])).

*Mentor: Patricia and Valentin*

Talk 11 - 15.12. (*hard*):

***Topic:*** **A Calabi quasimorphism on $\mathrm{Ham}(S^2)$.**
*References:* [EP03].
*Content:* Try to extract the general ideas/concepts for the key words listed below.

- Floer homology (section 2.5), Quantum homology (intro 2.3 and 2.3.1), its relation (sections 2.6.1-2.6.2, [PSS96, Theorem 4.1]).
- Spectral invariants (section 2.6.2), construction of Calabi quasimorphism on $\mathrm{Ham}(S^2)$ (Theorem 3.1, sections 3.4 and 3.5).
- If time allows: Mention the formula for autonomous Hamiltonian diffeomorphisms (Theorem 5.4), or its relation to Hofer's metric.

*Mentor:* Patricia

Talk 12 - 22.12. (*easy*):

***Topic:*** **Construction of the real numbers by quasimorphisms.**
*References:* [AC21]
*Content:*

- Definition of real numbers via "slopes"
- Explain what it has to do with quasimorphisms, define addition, multiplication, integers.
- What are rational numbers, $\sqrt{2}$, root of a polynomial, $\pi$ in this model?

*Mentor:* Patricia

## References


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