Prof. Wendelin Werner
Matthis Lehmkuehler HS 2021

SCHRAMM-LOEWNER EVOLUTIONS (D-MATH)
EXERCISE SHEET 1 – SOLUTION

Throughout this exercise sheet, let $\gamma \sim \mathrm{SLE}_\kappa$ for $\kappa > 0$ and let us write $\xi = \sqrt{\kappa} B$ for its Loewner driving function where B is a standard Brownian motion. Furthermore, let (g_t) be the mapping out functions, (K_t) the chordal hulls and (ζ_z) be the swallowing times.

Exercise 1. Suppose that $\kappa < 4$. Our aim will be to show that $|\gamma_t| \to \infty$ as $t \to \infty$ almost surely (this property is called transience of SLE).

(i) Show that $g_t(1) - \xi_t \to \infty$ as $t \to \infty$ almost surely.
(ii) Hence establish that $\inf_{t \geq 0} |\gamma_t - 1| > 0$ a.s.
(iii) Show that for all $x \in \mathbb{R} \setminus \{0\}$ we have $\inf_{t \geq 0} |\gamma_t - x| > 0$ a.s.
(iv) Let x_- (resp. x_+) be the left (resp. right) image of 0 under g_1, i.e. $x_\pm = \lim_{\epsilon \downarrow 0} g_1(\pm\epsilon)$. Argue that $\inf_{t \geq 1} |g_1(\gamma_t) - x_\pm| > 0$ almost surely.
(v) Deduce that $\inf_{t \geq 1} |\gamma_t| > 0$ almost surely and hence prove the transience of γ.

The result derived in this exercise holds in fact for all $\kappa > 0$ and for the rest of the exercise sheet you may use this result in the entire valid parameter range.

Solution. (i) We use Itô's formula to get that a.s. for all $t < \zeta_1$,

$$d(g_t(1) - \xi_t) = \frac{2\,dt}{g_t(1) - \xi_t} - \sqrt{\kappa}\,dB_t \quad \text{and hence}$$

$$d\left(\frac{g_t(1) - \xi_t}{\sqrt{\kappa}}\right) = \frac{\delta - 1}{2} \frac{dt}{g_t(1) - \xi_t} - dB_t$$

where $(\delta - 1)/2 = 2/\kappa$ i.e. $\delta = 1 + 4/\kappa$. The assumption $\kappa < 4$ implies that $\delta > 2$ and hence $((g_t(1) - \xi_t)/\sqrt{\kappa} \colon t < \zeta_1)$ is a Bessel process of dimension $\delta > 2$; in particular $\zeta_1 = \infty$ a.s. (this has already been shown in the lectures) and $(g_t(1) - \xi_t)/\sqrt{\kappa} \to \infty$ as $t \to \infty$ almost surely.

(ii) Define the stopping time $\tau_\epsilon = \inf\{t \geq 0 \colon |\gamma_t - 1| = \epsilon\}$. Our goal is now to show (roughly speaking) that on the event $\tau_\epsilon < \infty$ implies the quantity $g_{\tau_\epsilon}(z) - \xi_{\tau_\epsilon}$ is small; this will be a purely deterministic statement but we will use a planar Brownian motion to understand this deterministic phenomenon.

Let us now assume that $\epsilon \in (0, 1)$ and $\tau_\epsilon < \infty$ and we fix a realization $\gamma|_{[0,\tau_\epsilon]}$. If we start a planar Brownian motion B from $g_t^{-1}(iy)$ for $y > 0$ then if it exits $\mathbb{H} \setminus \gamma([0, t])$ by hitting the right side of $\gamma([0, t])$ or the interval $[0, 1]$ then it needs to hit $\partial B_\epsilon(1)$ before exiting $\mathbb{H}$ (draw a picture). By conformal invariance of planar Brownian motion therefore

$$\mathbb{P}_{iy}(B_{\nu_{\mathbb{H}}} \in [\xi_{\tau_\epsilon}, g_{\tau_\epsilon}(1)]) \leq \mathbb{P}_{g_{\tau_\epsilon}^{-1}(iy)}(B_{\nu_{\mathbb{H} \setminus \bar{B}_\epsilon(1)}} \in \partial B_\epsilon(1))$$

where $\nu_D = \inf\{t \geq 0 \colon B_t \notin D\}$ is the exit time from B. By multiplying both sides with y and letting $y \to \infty$ we obtain

$$g_{\tau_\epsilon}(1) - \xi_{\tau_\epsilon} \leq C\epsilon$$

for some universal constant $C > 0$. We deduce that $\tau_\epsilon = \infty$ provided that $\epsilon < \inf_{t \geq 0}(g_t(1) - \xi_t)/C$ and the result follows.

(iii) If $x > 0$ then by the scaling property of SLE,

$$\inf_{t\geq 0} |\gamma_t - x| \overset{d}{=} \inf_{t\geq 0} |x\gamma_{t/x^2} - x| = x \inf_{t\geq 0} |\gamma_t - 1| > 0 \quad \text{a.s.}$$

The claim for $x < 0$ follows from the reflection property of SLE i.e. that γ has the same law as $-\bar{\gamma}$ (which is the reflection of γ across the imaginary axis).

(iv) This follows from the Markov property of SLE: Indeed, $g_1(\gamma(\cdot + 1)) - \xi_1$ is an SLE_κ which is independent of $\gamma|_{[0,1]}$. In particular, the values $x_\pm - \xi_1$ are non-zero and independent of the SLE_κ given by $g_1(\gamma(\cdot + 1)) - \xi_1$. The claim now follows from (iii).

(v) Fix a realization of γ and suppose that $\delta > 0$ is such that $|g_1(\gamma_t) - x_+| \geq \delta$ and $|g_1(\gamma_t) - x_-| \geq \delta$ for all $t \geq 1$ (by (iv) this occurs almost surely). It follows that $\gamma([0,1]) \cup g_1^{-1}(B_\delta(x_+)) \cup g_1^{-1}(B_\delta(x_-))$ contains a semiball $B_{\delta'}(0) \cap \mathbb{H}$ for some $\delta' > 0$. In particular, $|\gamma_t| \geq \delta'$ for all $t \geq 1$ as required.

To conclude, note that if we let $S := \liminf_{t\to\infty} |\gamma_t|$ then we just established that $S > 0$ almost surely. However, by the scaling property of SLE, we have $S =_d \lambda S$ for all $\lambda > 0$ and it is easy to conclude that this implies $S = \infty$ almost surely.

Exercise 2. Fix $z \in \mathbb{H}$. Suppose that $\kappa < 8$ so that we have $z \notin \gamma([0,\infty))$ almost surely.

(i) Using Itô's formula, write down the decomposition of $(\log(g_t(z) - \xi_t) \colon t < \zeta_z)$ and hence of $(\arg(g_t(z) - \xi_t) \colon t < \zeta_z) = (\Im \log(g_t(z) - \xi_t) \colon t < \zeta_z)$ into a local martingale and a finite variation part.

(ii) Hence find the (unique) continuous function $f \colon [0,\pi] \to \mathbb{R}$ which is smooth on $(0,\pi)$ and satisfies $f(0) = 0$ and $f(\pi) = 1$ such that

$$M = (f(\arg(g_t(z) - \xi_t)) \colon t < \zeta_z)$$

is a local martingale.

(iii) Deduce that γ passes to the right of z with probability

$$\frac{\int_0^{\arg(z)} \sin(\theta)^{8/\kappa - 2}\, d\theta}{\int_0^{\pi} \sin(\theta)^{8/\kappa - 2}\, d\theta} .$$

Hint: Use the optional stopping theorem and exercise 1.

Solution. (i) Almost surely for $t < \zeta_z$,

$$\begin{aligned}
d\log(g_t(z) - \xi_t) &= \frac{1}{g_t(z) - \xi_t}\left(\frac{2\,dt}{g_t(z) - \xi_t} - \sqrt{\kappa}\,dB_t\right) - \frac{\kappa\,dt}{2(g_t(z) - \xi_t)^2} \\
&= \frac{2 - \kappa/2}{(g_t(z) - \xi_t)^2}\,dt - \frac{\sqrt{\kappa}\,dB_t}{g_t(z) - \xi_t} , \\
d\arg(g_t(z) - \xi_t) &= d\Im\log(g_t(z) - \xi_t) \\
&= (2 - \kappa/2)\Im\left(\frac{1}{(g_t(z) - \xi_t)^2}\right)dt - \Im\left(\frac{1}{g_t(z) - \xi_t}\right)\sqrt{\kappa}\,dB_t .
\end{aligned}$$

This yields the desired decomposition.

(ii) Suppose that $f \colon [0,\pi] \to \mathbb{R}$ is smooth on $(0,\pi)$. Let us write $A_t = \arg(g_t(z) - \xi_t)$ and $D_t = |g_t(z) - \xi_t|$. Then by Itô's formula, almost surely for all $t < \zeta_z$,

$$\begin{aligned}
df(A_t) &= f'(A_t)\,dA_t + \frac{1}{2}f''(A_t)\,d\langle\theta\rangle_t \\
&= f'(A_t)(2 - \kappa/2)\Im(D_t^{-2}e^{-2iA_t})\,dt - f'(A_t)\sqrt{\kappa}\Im(D_t^{-1}e^{-iA_t})\,dB_t \\
&\quad + \frac{\kappa}{2}f''(A_t)(\Im(D_t^{-1}e^{-iA_t}))^2\,dt .
\end{aligned}$$

Thus f is a local martingale on $(0, \zeta_z)$ if and only if for all $x \in (0, \pi)$,

$$f'(x)(2 - \kappa/2)\sin(-2x) + \frac{\kappa}{2} f''(x)\sin^2(x) = 0 .$$

To solve this ODE, we first set $g = f'$ and write everything as

$$\frac{d}{dx}\log g(x) = (8/\kappa - 2)\frac{d}{dx}\log\sin(x)$$

Hence $g(x) = C\sin(x)^{8/\kappa-2}$ for $C \in \mathbb{R}$. Thus for some $B \in \mathbb{R}$,

$$f(x) = B + C\int_0^x \sin(\theta)^{8/\kappa-2}\, d\theta .$$

We pick $B = 0$ and $C = (\int_0^\pi \sin(\theta)^{8/\kappa-2}\, d\theta)^{-1}$ so that $f(0) = 0$ and $f(\pi) = 1$.

(iii) Let us work on the almost sure event where $z \notin \gamma([0,t])$ and where $|\gamma_t| \to \infty$ as $t \to \infty$. The key observation to answer the question is that $\arg(g_t(z) - \xi_t) \to \pi 1_{P_R}$ as $t \uparrow \zeta_z$ where P_R is the event that γ passes to the right of z. This is easy to see by using a planar Brownian motion B. Indeed, let us fix a realization of γ. Note that since the argument is a harmonic and bounded function, we have for $t < \zeta_z$,

$$\arg(g_t(z) - \xi_t)/\pi = \mathbb{P}_{g_t(z)-\xi_t}(B_{\nu_{\mathbb{H}}} \in (-\infty, 0)) .$$

However, by conformal invariance of Brownian motion, the right hand side is the probability that a Brownian motion started from z exits the domain $\mathbb{H} \setminus K_t$ by hitting the boundary of this domain which is located right of its γ_t. We see that this probability tends to 1 as $t \uparrow \zeta_z$ if γ passes right of z and tends to 0 otherwise (almost surely).

Hence, if we define $M_t = \pi 1_{P_R}$ for $t \in [\zeta_z, \infty]$ then M defines a bounded martingale on $[0, \infty]$ (note that if $\kappa \leq 4$ we are only defining a new value at ∞) and optional stopping at time t yields the result.

Exercise 3. Suppose that $\kappa = 4$ and fix $z \in \mathbb{H}$. Recall that $\zeta_z = \infty$ a.s. Let P_L and P_R be the events that γ passes to the left of z and to the right of z respectively. We also let $R_\infty = R(z, \mathbb{H} \setminus \gamma([0,\infty)))$ be the conformal radius of z in the complement of γ.

(i) Write down $\mathbb{P}(P_R)$ in terms of $\arg(z)$.

(ii) Show that

$$R_t := \Im g_t(z)/|g_t'(z)| = R(z, \mathbb{H} \setminus \gamma([0,t])) \to R_\infty \quad \text{and}$$
$$A_t := \Im \log(g_t(z) - \xi_t) = \arg(g_t(z) - \xi_t) \to \pi 1_{P_R}$$

as $t \to \infty$ a.s.

(iii) For $\theta \in \mathbb{C}$ show that M is a local martingale where

$$M_t = e^{\theta A_t} R_t^{\theta^2/2}$$

Hint: First apply Itô's formula to A and $\log R$ and then exponentiate.

(iv) Using optional stopping and suitable parameter choices show that for $\theta > 0$,

$$\mathbb{E}(R_\infty^{\theta^2/2} \mid P_R) = (\Im z)^{\theta^2/2}\, \frac{\pi}{\arg(z)} \cdot \frac{\sinh(\theta\arg(z)))}{\sinh(\theta\pi)} .$$

Solution. (i) By exercise 2 we know that $\mathbb{P}(P_R) = \arg(z)/\pi$.

(ii) The statement about the argument process has already been argued in exercise 2 part (iii), so it suffices to prove the statement about the conformal radii. The argument is going to be quite similar to the one pertaining to the argument. Let $\phi_t\colon \mathbb{D} \to \mathbb{H} \setminus K_t$ be a conformal transformation with $\phi_t(0) = z$; by the definition of conformal radii,

$$R_t = R(z, \mathbb{H} \setminus K_t) = |\phi_t'(0)| .$$

Let $\tilde{\gamma}_t = \phi_t^{-1}(\gamma(\cdot + t))$ (draw a figure). The curve $\tilde{\gamma}_t$ cuts the unit disk $\mathbb{D}$ into two disjoint simply connected domains and we write D_t for the one containing 0. Note that

$$R_\infty = R(z, \mathbb{H} \setminus \gamma([0,\infty))) = R_t \cdot R(0, D_t) .$$

Thus it suffices to show that $R(0, D_t) \to 1$ as $t \to \infty$ almost surely. For this, we again fix a realization of γ and use a planar Brownian motion B as a tool to understand some complex analysis distortion estimates. Let p_t be the probability that the Brownian motion B started from z exits the domain $\mathbb{H} \setminus K_t$ on the union of the right boundary of $\gamma([0,t])$ with $(0,\infty)$ (resp. the left boundary of $\gamma([0,t])$ with $(-\infty, 0)$) if P_R (resp. P_L) holds. By the conformal invariance of planar Brownian motion, we have

$$p_t = \mathbb{P}_0(\nu_{D_t} \notin \mathbb{D})$$

where $\nu_D = \inf\{t \geq 0\colon B_t \notin D\}$ is the exit time from a domain D. Let us define $d_t = \sup\{|z|\colon z \in D_t\}$. Then it is not hard to see that $\mathbb{P}_0(\nu_{D_t} \notin \mathbb{D}) \geq C(1 - d_t)$ for some universal constant. Moreover, $R(0, D_t) \geq R(0, d_t\mathbb{D}) = d_t$. We can now conclude since $p_t \to 0$ as $t \to \infty$ a.s. by transience of SLE.

(iii) Let $D_t = |g_t(z) - \xi_t|$. In exercise 2 we already computed that (noting that we have $\kappa = 4$ now): Almost surely for all $t < \zeta_z$,

$$dA_t = -2 \cdot \Im\left(\frac{1}{g_t(z) - \xi_t}\right) dB_t = \frac{2\sin(A_t)}{D_t}\, dB_t .$$

Since both $(g_t(z))$ and $(g_t'(z))$ are differentiable functions, we have that a.s. for $t < \zeta_z$,

$$\begin{aligned}
d\log \Im(g_t(z)) &= \frac{1}{\Im g_t(z)} \Im\left(\frac{2}{g_t(z) - \xi_t}\right) dt = \frac{\Im(2D_t^{-1}e^{-iA_t})}{\Im(D_t e^{iA_t})}\, dt = \frac{-2}{D_t^2}\, dt \\
d\log g_t'(z) &= \frac{1}{g_t'(z)} \frac{d}{dz}\left(\frac{2}{g_t(z) - \xi_t}\right) dt = \frac{-2\, dt}{(g_t(z) - \xi_t)^2} , \\
d\log |g_t'(z)| &= d\Re \log g_t'(z) = \Re\left(\frac{-2}{(g_t(z) - \xi_t)^2}\right) dt = \frac{-2\cos(2A_t)}{D_t^2}\, dt
\end{aligned}$$

Therefore, we obtain that a.s. for $t < \zeta_z$,

$$\begin{aligned}
d\left(\theta A_t + \frac{\theta^2}{2}\log R_t\right) &= \frac{2\theta\sin(A_t)}{D_t}\, dB_t - \frac{\theta^2}{D_t^2}\, dt + \frac{\theta^2\cos(2A_t)}{D_t^2}\, dt \\
&= \frac{2\theta\sin(A_t)}{D_t}\, dB_t - \frac{2\theta^2\sin^2(A_t)}{D_t^2}\, dt .
\end{aligned}$$

From this, we can directly see that M is a local martingale since it is the exponential (local) martingale associated the process above.

(iv) Note that A is a process taking values in $(0, \pi)$ and R is a non-decreasing process (in particular $R_t \leq R_0 = \Im z$ for all $t \geq 0$). Thus M is not just a local martingale but in fact a bounded martingale and by part (ii) we may use optional stopping to deduce

$$e^{\theta \arg(z)}(\Im z)^{\theta^2/2} = \mathbb{E}(R_\infty^{\theta^2/2} 1_{P_L} + e^{\theta\pi} R_\infty^{\theta^2/2} 1_{P_R}) \quad \text{and}$$
$$e^{-\theta \arg(z)}(\Im z)^{\theta^2/2} = \mathbb{E}(R_\infty^{\theta^2/2} 1_{P_L} + e^{-\theta\pi} R_\infty^{\theta^2/2} 1_{P_R}) \ .$$

The result follows by taking a suitable linear combination and using part (i).

Exercise 4. Suppose now that $\kappa > 8$. The goal of this question will be to show that $\gamma([0,\infty)) = \overline{\mathbb{H}}$ almost surely.

(i) Show using exercise 1 that it suffices to prove that $z \in \gamma([0,\infty))$ a.s. for all $z \in \mathbb{H}$.

(ii) We now fix $z \in \mathbb{H}$. Show that for each fixed $\rho \in \mathbb{R}$ the process M is a local martingale on $(0, \zeta_z)$ where

$$M_t = |g_t'(z)|^{(8-2\kappa+\rho)\rho/(8\kappa)} (\Im g_t(z))^{\rho^2/(8\kappa)} |g_t(z) - \xi_t|^{\rho/\kappa} 1(t < \zeta_z) \ .$$

Hint: It will be useful to first apply Itô's formula to

$$Z_t = \frac{(8-2\kappa+\rho)\rho}{8\kappa} \log(g_t'(z)) + \frac{\rho^2}{8\kappa} \log(\Im g_t(z)) + \frac{\rho}{\kappa} \log(g_t(z) - \xi_t) \ ,$$

take the real part of the resulting expression and exponentiate.

(iii) Deduce that M is a supermartingale.

(iv) We will now restrict to the case $\rho = \kappa - 8$. Show that in this case

$$M_t = \frac{R(z, \mathbb{H} \setminus K_t)^{\kappa/8-1}}{\sin(\arg(g_t(z) - \xi_t))^{1-8/\kappa}} 1(t < \zeta_z) \ .$$

where $R(z, \mathbb{H} \setminus K_t) = \Im g_t(z)/|g_t'(z)|$ denotes the conformal radius of z in the complement of the hull K_t.

(v) Argue using exercise 1 that $z \in \gamma([0,\infty))$ almost surely. Hint: Use the Koebe 1/4 theorem to compare the conformal radius with a Euclidean distance.

The result of the exercise remains true for $\kappa = 8$ but the proof relies on the convergence of the Uniform Spanning Tree Peano Curve to SLE_8 (in distribution).

Solution. (i) By exercise 1 and the assumption, almost surely, $|\gamma_t| \to \infty$ as $t \to \infty$ and $q \in \gamma([0,t])$ for all $q \in \mathbb{Q} + i\mathbb{Q}_{>0}$. Let us work on this event and consider any $z \in \bar{\mathbb{H}}$. Then there exists $q_n \in \mathbb{Q} + i\mathbb{Q}_{>0}$ and $t_n \geq 0$ such that $\gamma(t_n) = q_n \to z$ as $n \to \infty$. Moreover, there exists a $T \geq 0$ such that $|\gamma(t)| \geq |z| + 1$ for all $t \geq T$; in particular $t_n \leq T$ for all $n \geq 1$. We may therefore extract a convergent subsequence $t_{n_k} \to t_*$ as $k \to \infty$ and we have $\gamma(t_*) = z$ by continuity of γ.

(ii) Let $A_t = \arg(g_t(z) - \xi_t)$ and $D_t = |g_t(z) - \xi_t|$ for $t < \zeta_z$. By several applications of Itô's formula, we get that a.s. for all $t < \zeta_z$,

$$d \log g_t'(z) = \frac{1}{g_t'(z)} \frac{d}{dz} \left(\frac{2}{g_t(z) - \xi_t} \right) dt = \frac{-2\, dt}{(g_t(z) - \xi_t)^2} = \frac{-2\, dt}{D_t^2 e^{2iA_t}} \ ,$$
$$d \log \Im(g_t(z)) = \frac{1}{\Im g_t(z)} \Im \left(\frac{2}{g_t(z) - \xi_t} \right) dt = \frac{\Im(2 D_t^{-1} e^{-iA_t})}{\Im(D_t e^{iA_t})} dt = \frac{-2\, dt}{D_t^2}$$
$$d \log(g_t(z) - \xi_t) = \frac{1}{g_t(z) - \xi_t} \left(\frac{2\, dt}{g_t(z) - \xi_t} - \sqrt{\kappa}\, dB_t \right) - \frac{\kappa\, dt}{2(g_t(z) - \xi_t)^2}$$
$$= \frac{(2 - \kappa/2)\, dt}{D_t^2 e^{2iA_t}} - \frac{\sqrt{\kappa}\, dB_t}{D_t e^{iA_t}} \ .$$

Taking real parts and a linear combination yields that a.s. for $t < \zeta_z$,

$$\begin{aligned} d\Re Z_t &= \frac{(8-2\kappa+\rho)\rho}{8\kappa} \cdot \frac{-2\cos(2A_t)\,dt}{D_t^2} + \frac{\rho^2}{8\kappa} \cdot \frac{-2\,dt}{D_t^2} \\ &\quad + \frac{\rho}{\kappa}\left(\frac{(2-\kappa/2)\cos(2A_t)\,dt}{D_t^2} - \frac{\sqrt{\kappa}\cos(A_t)\,dB_t}{D_t}\right) \\ &= \frac{-\rho^2\cos^2(A_t)}{2\kappa D_t^2}\,dt - \frac{\rho\cos(A_t)}{\sqrt{\kappa}D_t}\,dB_t\ . \end{aligned}$$

From this, we immediately deduce that $M = e^{\Re Z}$ is a local martingale on $(0, \zeta_z)$.

(iii) The fact that M is a supermartingale is immediate since any non-negative local martingale is always a supermartingale.

(iv) We have $R(z, \mathbb{H} \setminus K_t) = \Im g_t(z)/|g_t'(z)|$ and $\sin(\arg(g_t(z)-\xi_t)) = \Im g_t(z)/|g_t(z)-\xi_t|$. The statement is now just a matter of rearranging quantities.

(v) By (iii) we know that M has a càdlàg version and we will switch to such a version. Let us also work on the (almost sure) event where $|\gamma_t| \to \infty$ as $t \to \infty$. By the Koebe 1/4 theorem, we know that for $t < \zeta_z$

$$\Im g_t(z)/|g_t'(z)| \geq \frac{\mathrm{dist}(z, K_t \cup \mathbb{R})}{4}\ .$$

We deduce that

$$M_t \geq \left(\frac{\mathrm{dist}(z, \gamma([0,\infty)) \cup \mathbb{R})}{4}\right)^{\kappa/8-1} \frac{1}{\sin(\arg(g_t(z)-\xi_t))^{1-8/\kappa}}\, 1(t < \zeta_z)$$

On the event where $z \notin \gamma([0,\infty))$ the same argument as in exercise 2 part (iii) implies that we have $\sin(\arg(g_t(z) - \xi_t)) \to 0$ as $t \uparrow \zeta_z$ and moreover the first term in the last display is strictly positive contradicting the existence of a left limit at ζ_z of M. Therefore $z \in \gamma([0,\infty))$ almost surely.