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SCHRAMM-LOEWNER EVOLUTIONS (D-MATH)
EXERCISE SHEET 1

Throughout this exercise sheet, let $\gamma \sim \text{SLE}_\kappa$ for $\kappa > 0$ and let us write $\xi = \sqrt{\kappa} B$ for its Loewner driving function where B is a standard Brownian motion. Furthermore, let (g_t) be the mapping out functions, (K_t) the chordal hulls and (ζ_z) be the swallowing times.

Exercise 1. Suppose that $\kappa < 4$. Our aim will be to show that $|\gamma_t| \to \infty$ as $t \to \infty$ almost surely (this property is called transience of SLE).

(i) Show that $g_t(1) - \xi_t \to \infty$ as $t \to \infty$ almost surely.
(ii) Hence establish that $\inf_{t \geq 0} |\gamma_t - 1| > 0$ a.s.
(iii) Show that for all $x \in \mathbb{R} \setminus \{0\}$ we have $\inf_{t \geq 0} |\gamma_t - x| > 0$ a.s.
(iv) Let x_- (resp. x_+) be the left (resp. right) image of 0 under g_1, i.e. $x_\pm = \lim_{\epsilon \downarrow 0} g_1(\pm\epsilon)$. Argue that $\inf_{t \geq 1} |g_1(\gamma_t) - x_\pm| > 0$ almost surely.
(v) Deduce that $\inf_{t \geq 1} |\gamma_t| > 0$ almost surely and hence prove the transience of γ.

The result derived in this exercise holds in fact for all $\kappa > 0$ and for the rest of the exercise sheet you may use this result in the entire valid parameter range.

Exercise 2. Fix $z \in \mathbb{H}$. Suppose that $\kappa < 8$ so that we have $z \notin \gamma([0, \infty))$ almost surely.

(i) Using Itô's formula, write down the decomposition of $(\log(g_t(z) - \xi_t) \colon t < \zeta_z)$ and hence of $(\arg(g_t(z) - \xi_t) \colon t < \zeta_z) = (\Im \log(g_t(z) - \xi_t) \colon t < \zeta_z)$ into a local martingale and a finite variation part.
(ii) Hence find the (unique) continuous function $f \colon [0, \pi] \to \mathbb{R}$ which is smooth on $(0, \pi)$ and satisfies $f(0) = 0$ and $f(\pi) = 1$ such that

$$M = (f(\arg(g_t(z) - \xi_t)) \colon t < \zeta_z)$$

is a local martingale.
(iii) Deduce that γ passes to the right of z with probability

$$\frac{\int_0^{\arg(z)} \sin(\theta)^{8/\kappa - 2} \, d\theta}{\int_0^{\pi} \sin(\theta)^{8/\kappa - 2} \, d\theta} .$$

Hint: Use the optional stopping theorem and exercise 1.

Exercise 3. Suppose that $\kappa = 4$ and fix $z \in \mathbb{H}$. Recall that $\zeta_z = \infty$ a.s. Let P_L and P_R be the events that γ passes to the left of z and to the right of z respectively. We also let $R_\infty = R(z, \mathbb{H} \setminus \gamma([0,\infty)))$ be the conformal radius of z in the complement of γ.

(i) Write down $\mathbb{P}(P_R)$ in terms of $\arg(z)$.

(ii) Show that

$$R_t := \Im g_t(z)/|g_t'(z)| = R(z, \mathbb{H} \setminus \gamma([0,t])) \to R_\infty \quad \text{and}$$
$$A_t := \Im \log(g_t(z) - \xi_t) = \arg(g_t(z) - \xi_t) \to \pi 1_{P_R}$$

as $t \to \infty$ a.s.

(iii) For $\theta \in \mathbb{C}$ show that M is a local martingale where

$$M_t = e^{\theta A_t} R_t^{\theta^2/2}$$

Hint: First apply Itô's formula to A and $\log R$ and then exponentiate.

(iv) Using optional stopping and suitable parameter choices show that for $\theta > 0$,

$$\mathbb{E}(R_\infty^{\theta^2/2} \mid P_R) = (\Im z)^{\theta^2/2} \frac{\pi}{\arg(z)} \cdot \frac{\sinh(\theta \arg(z)))}{\sinh(\theta\pi)} .$$

Exercise 4. Suppose now that $\kappa > 8$. The goal of this question will be to show that $\gamma([0,\infty)) = \overline{\mathbb{H}}$ almost surely.

(i) Show using exercise 1 that it suffices to prove that $z \in \gamma([0,\infty))$ a.s. for all $z \in \mathbb{H}$.

(ii) We now fix $z \in \mathbb{H}$. Show that for each fixed $\rho \in \mathbb{R}$ the process M is a local martingale on $(0, \zeta_z)$ where

$$M_t = |g_t'(z)|^{(8-2\kappa+\rho)\rho/(8\kappa)} (\Im g_t(z))^{\rho^2/(8\kappa)} |g_t(z) - \xi_t|^{\rho/\kappa} 1(t < \zeta_z) .$$

Hint: It will be useful to first apply Itô's formula to

$$Z_t = \frac{(8 - 2\kappa + \rho)\rho}{8\kappa} \log(g_t'(z)) + \frac{\rho^2}{8\kappa} \log(\Im g_t(z)) + \frac{\rho}{\kappa} \log(g_t(z) - \xi_t) ,$$

take the real part of the resulting expression and exponentiate.

(iii) Deduce that M is a supermartingale.

(iv) We will now restrict to the case $\rho = \kappa - 8$. Show that in this case

$$M_t = \frac{R(z, \mathbb{H} \setminus K_t)^{\kappa/8-1}}{\sin(\arg(g_t(z) - \xi_t))^{1-8/\kappa}} 1(t < \zeta_z) .$$

where $R(z, \mathbb{H} \setminus K_t) = \Im g_t(z)/|g_t'(z)|$ denotes the conformal radius of z in the complement of the hull K_t.

(v) Argue using exercise 1 that $z \in \gamma([0,\infty))$ almost surely. Hint: Use the Koebe 1/4 theorem to compare the conformal radius with a Euclidean distance.

The result of the exercise remains true for $\kappa = 8$ but the proof relies on the convergence of the Uniform Spanning Tree Peano Curve to SLE_8 (in distribution).

Submission of solutions. Send your solutions via email to Matthis Lehmkuehler by Monday 01/11/21 at 5 p.m.