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SCHRAMM-LOEWNER EVOLUTIONS (D-MATH)
EXERCISE SHEET 2 – SOLUTION

Throughout this exercise sheet, whenever γ is a curve, we write ξ for its driving function. Furthermore, let (g_t) be the mapping out functions, (K_t) the chordal hulls and (ζ_z) be the swallowing times.

Exercise 1. The goal of this question will be to prove a characterization result for squared Bessel processes. We suppose that X^x is a continuous process started from x and taking values in $(0, \infty)$ for each $x > 0$. We assume that the family (X^x) defines a Markov process and that it satisfies the scaling property

$$X^x \overset{d}{=} (\lambda X^{x/\lambda}_{t/\lambda} \colon t \geq 0) \quad \text{for all } \lambda, x > 0 \ .$$

Our goal will be to show that (X^x) is (up to rescaling) a squared Bessel process of dimension $\delta \geq 2$.

(i) Show that (X^x) defines a strong Markov process.
(ii) For $x > 0$ let

$$\begin{aligned} \sigma^x_t &= \int_0^t du / X^x_u \quad \text{for } t \geq 0 \ , \\ \tau^x_s &= \inf \{ t \geq 0 \colon \sigma^x_t \geq s \} \quad \text{for } s \geq 0 \ , \\ P^x &= \log(X^x \circ \tau^x) \ . \end{aligned}$$

Show that the process P^x has independent and stationary increments.
(iii) Deduce that there are constants $\mu \in \mathbb{R}$ and $\sigma \geq 0$ such that

$$P^x = (\log(x) + \mu s + \sigma W^x_s \colon s \geq 0) \quad \text{for all } x > 0$$

where W^x is a standard Brownian motion for each $x > 0$.
(iv) Deduce that X^x satisfies the SDE

$$\begin{aligned} dX^x_t &= (\mu + \sigma^2/2) \, dt + \sigma \sqrt{X^x_t} \, dB^x_t \quad \text{where} \\ B^x &:= \int_0^\cdot \sqrt{X^x_u} \, d(W^x \circ \sigma^x)_u \quad \text{is a standard Brownian motion .} \end{aligned}$$

(v) If $\sigma = 0$ then X^x is just a deterministic affine function and so we suppose that $\sigma > 0$. Let $\delta = 2 + 4\mu/\sigma^2$ and $\lambda = 4/\sigma^2$. Show that $\delta \geq 2$ and that λX^x is a squared Bessel process of dimension δ started from x.

Solution. (i) To see that the strong Markov property holds, since we know that X^x is continuous it suffices to show that

$$x \mapsto \mathbb{E}(f(X^x_t))$$

is continuous on $(0, \infty)$ for all $t \geq 0$ and whenever $f \colon (0, \infty) \to \mathbb{R}$ is bounded and continuous. But by the scaling property we have that $\mathbb{E}(f(X^x_t)) = \mathbb{E}(f(x X^1_{t/x}))$ and the claim is then clear by the continuity of X^1 (and dominated convergence).

(ii) Using the scaling property of the family (X^x) we have for $\lambda, x > 0$ that

$$P^x \stackrel{d}{=} \left(\log(\lambda) + \log X^{x/\lambda}_{\lambda^{-1}\inf\left\{t\geq 0:\ \int_0^t du/(\lambda X^{x/\lambda}_{u/\lambda})\geq s\right\}} : s \geq 0\right)$$
$$= \left(\log(\lambda) + \log X^{x/\lambda}_{\inf\left\{t\geq 0:\ \int_0^t du/X^{x/\lambda}_u\geq s\right\}} : s \geq 0\right) = \log(\lambda) + P^{x/\lambda} .$$

Moreover, for any fixed $s_0 \geq 0$ we can write

$$(P^x_{s+s_0} : s \geq 0) = \left(X^x_{\inf\left\{t\geq 0:\ \int_0^t du/X^x_u\geq s+s_0\right\}} : s \geq 0\right)$$
$$= \left(X^x_{\tau^x_{s_0}+\inf\left\{t\geq 0:\ \int_0^t du/X^x_{u+\tau^x_{s_0}}\geq s\right\}} : s \geq 0\right) .$$

So by the strong Markov property we have that the conditional law of $(P^x_{s+s_0} : s \geq 0)$ given $(X^x)^{\tau^x_{s_0}}$ is the law of P^y with

$$y = \log X^x_{\tau^x_{s_0}} = P^x_{s_0}$$

which is the same (by the previous computation) as the law of $P^x + \log(y/x)$. Since $P^x|_{[0,s_0]}$ is measurable with respect to $(X^x)^{\tau^x_{s_0}}$ we get that the conditional law of $(P^x_{s_0+s} - P_{s_0} : s \geq 0)$ given $P^x_{[0,s_0]}$ is the law of P^1 and the stationarity and independence of the increments of the process P^x follows.

(iii) Any process with stationary and independent increments is a Brownian motion with drift so by (ii) there exists $\mu \in \mathbb{R}$ and $\sigma \geq 0$ and a standard Brownian motion W^1 such that the claim holds for $x = 1$. The result for general $x > 0$ follows from the scaling property of P^x derived in (ii).

(iv) Note that $X^x = \exp(P^x \circ \sigma^x)$. Let us first analyse the process B^x. We have for a.s. for all $t \geq 0$ that

$$B^x_t = \int_0^{\sigma^x_t} \sqrt{X^x_{\tau^x_v}}\, dW^x_v ,$$
$$\langle B^x \rangle_t = \int_0^{\sigma^x_t} X^x_{\tau^x_v}\, dv = \int_0^t X^x_u\, d\sigma^x_u = t .$$

Note that the first line implies that B^x is a local martingale and by Lévy's characterisation and the second line we get that B^x is a standard Brownian motion. For all for $t \geq 1$ we have almost surely

$$P^x_{\sigma^x_t} = \log(x) + \mu\sigma^x_t + \sigma W^x_{\sigma^x_t} = \log(x) + \mu \int_0^t \frac{du}{X^x_u} + \sigma \int_0^t \frac{dB^x_u}{\sqrt{X^x_u}} .$$

Therefore by Itô's formula we get that a.s. for $t \geq 0$,

$$dX^x_t = X^x_t \left(d(P^x \circ \sigma_x)_t + \frac{1}{2}\, d\langle P^x \circ \sigma^x \rangle_t \right)$$
$$= X^x_t \left(\frac{\mu\, dt}{X^x_t} + \frac{\sigma\, dB^x_t}{\sqrt{X^x_t}} + \frac{\sigma^2\, dt}{2X^x_t} \right) = (\mu + \sigma^2/2)\, dt + \sigma \sqrt{X^x_t}\, dB^x_t .$$

Note that the fact that B^x is a standard Brownian motion simply follows from the fact that it is a local martingale and one can easily compute its quadratic variation process.

(v) By the definition of λ and δ we obtain from (iv) the SDE

$$d(\lambda X_t^x) = \delta\, dt + 2\sqrt{\lambda X_t^x}\, dB_t^x \ .$$

This is precisely the Bessel SDE of dimension δ. Finally, note that $\delta \geq 2$ since we assumed that X^x takes values in $(0,\infty)$, i.e. it does not hit 0.

Exercise 2. In this exercise, we will classify certain conformally invariant random curves. Let γ be a random curve starting at 0 generated by a Loewner chain with driving function ξ such that $\gamma([0,\infty)) \cap (-\infty,-1] = \emptyset$. Let $O_t = g_t(-1)$ which we call a marked point. We now assume that for $t \geq 0$ conditionally on $\gamma|_{[0,t]}$, the curve

$$\left(\frac{g_t(\gamma_{t+(\xi_t-O_t)^2 s}) - \xi_t}{\xi_t - O_t} \colon s \geq 0\right)$$

has the same law as γ. This is a conformal Markov property with a marked point. Note that the time rescaling factor $(\xi_t - O_t)^2$ appears only to ensure that the curve is parameterized by halfplane capacity. We also suppose that γ is not a deterministic curve.

(i) Let $Y = (\xi - O)^2$. Show that Y has the property that for $t \geq 0$ conditionally on $Y|_{[0,t]}$ the process $(Y_{t+Y_t s}/Y_t \colon s \geq 0)$ has the same law as Y.

(ii) Use exercise 1 to show that there exists $\delta \geq 2$ and $\kappa > 0$ such that $Y = \kappa X$ where X is a squared Bessel process starting from $1/\kappa$ and satisfying the SDE

$$dX_t = \delta\, dt + 2\sqrt{X_t}\, dB_t$$

where B is a standard Brownian motion.

(iii) Deduce that (ξ, O) satisfy the following system of SDEs

$$d\xi_t = \frac{\rho}{\xi_t - O_t}\, dt + \sqrt{\kappa}\, dB_t \ ,$$
$$dO_t = \frac{2\, dt}{O_t - \xi_t}$$

where $\rho = (\delta - 1)\kappa/2 - 2\,(\geq \kappa/2 - 2)$.

It turns out that the solution to the SDE in part (iii) indeed generates a continuous curve the law of which we call $\mathrm{SLE}_\kappa(\rho)$; in fact $\mathrm{SLE}_\kappa(\rho)$ can be defined whenever $\rho > -2$.

Solution. (i) Fix $t \geq 0$. The family of mapping out functions $(\tilde{g}_s)$ and the driving function $\tilde{\xi}$ of the curve

$$\left(\frac{g_t(\gamma_{t+(\xi_t-O_t)^2 s}) - \xi_t}{\xi_t - O_t} \colon s \geq 0\right) \quad \text{are given by}$$

$$\tilde{g}_s(z) = \frac{g_{t+(\xi_t-O_t)^2 s}(g_t^{-1}((\xi_t - O_t)z + \xi_t)) - \xi_t}{\xi_t - O_t} \ ,$$
$$\tilde{\xi}_s = \frac{\xi_{t+(\xi_t-O_t)^2 s} - \xi_t}{\xi_t - O_t} \ .$$

Therefore by our assumption, conditionally on $\xi|_{[0,t]}$ we have that

$$\left(\frac{Y_{t+Y_t s}}{Y_t} \colon s \geq 0\right) = \left(\frac{(\xi_{t+(\xi_t-O_t)^2 s} - O_{t+(\xi_t-O_t)^2 s})^2}{(\xi_t - O_t)^2} \colon s \geq 0\right) = \left((\tilde{\xi}_s - \tilde{g}_s(-1))^2 \colon s \geq 0\right)$$

has the same law as Y (since conditionally on $\xi|_{[0,t]}$ the process $\tilde{\xi}$ has the same law as ξ). Since $Y|_{[0,t]}$ is measurable with respect to $\xi|_{[0,t]}$ the claim follows.

(ii) By assumption, the point -1 is never swallowed and hence Y takes values in $(0,\infty)$. Furthermore the process Y determines $\xi - O$ and hence it also determines O – indeed this follows from $O_t = g_t(-1)$ and Loewner's equation; combining this yields that ξ is a deterministic function of Y and so the fact that ξ is not a deterministic function implies that Y is not a deterministic function. Therefore by the Markov property derived in (i) and exercise 1 we obtain the claim.

(iii) This is now just an exercise in stochastic calculus. Note that the second part of the system of SDEs follows from Loewner's equation since $O_t = g_t(-1)$. Moreover by Itô's formula a.s. for $t \geq 0$

$$\begin{aligned} d(\xi_t - O_t) = d\sqrt{Y_t} = \sqrt{\kappa}\, d\sqrt{X_t} = \sqrt{\kappa}\left(\frac{dX_t}{2\sqrt{X_t}} - \frac{d\langle X\rangle_t}{8(X_t)^{3/2}}\right) \\ = \frac{\delta-1}{2\sqrt{X_t}}\sqrt{\kappa}\, dt + \sqrt{\kappa}\, dB_t = \frac{(\delta-1)\kappa/2}{\xi_t - O_t}\, dt + \sqrt{\kappa}\, dB_t\ . \end{aligned}$$

Note that the application of Itô's formula is justified since the function $x \mapsto \sqrt{x}$ is twice differentiable on $(0,\infty)$ and the process X takes values in $(0,\infty)$.

Exercise 3. Fix $\alpha > 0$. Whenever K is a compact chordal hull satisfying $0 \notin K$ we write $\Phi_K \colon \mathbb{H} \setminus K \to \mathbb{H}$ for the unique conformal transformation with $\Phi_K(0) = 0$ and $\Phi_K(z)/z \to 1$ as $|z| \to \infty$. Let E be a curve from 0 to ∞ in $\overline{\mathbb{H}}$ satisfying

$$\mathbb{P}(E([0,\infty)) \cap K = \emptyset) = \Phi_K'(0)^\alpha \quad \text{for all chordal hulls with } 0 \notin K\ .$$

Also define $E' = (|\Re(E)| + i\Im(E))^2$ and let E'' be an independent copy of E'.

(i) Fix a compact chordal hull A such that $A \cap (-\infty, 0] = \emptyset$. Show that

$$\mathbb{P}(E'([0,\infty)) \cap A = \emptyset) = \mathbb{P}(E([0,\infty)) \cap (\sqrt{A} \cup A') = \emptyset) = \Phi'_{\sqrt{A}\cup A'}(0)^\alpha$$

where A' denotes the reflection of $\sqrt{A}$ across the imaginary axis.

(ii) Show that $\Phi_{\sqrt{A}\cup A'}(\epsilon) = \Phi_A(\epsilon^2)^{1/2}$ for $\epsilon > 0$ sufficiently small and deduce that

$$\mathbb{P}(E'([0,\infty)) \cap A = \emptyset) = \Phi_A'(0)^{\alpha/2}\ .$$

Hint: Use Schwarz reflection to write down $\Phi_{\sqrt{A}\cup A'}$ in terms of Φ_A.

(iii) Show that the right boundary of the set $E'([0,\infty)) \cup E''([0,\infty))$ has the same law as the right boundary of $E([0,\infty))$.

In the special case where E is a Brownian excursion from 0 to ∞ in $\mathbb{H}$ we have $\alpha = 1$ and the process E' is called a Brownian excursion from 0 to ∞ in $\mathbb{H}$ with perpendicular reflection along $(-\infty, 0)$.

Solution. (i) We have $E'_t \in A$ if and only if $|\Re(E_t)| + i\Im(E_t) \in \sqrt{A}$ which holds if and only if $E_t \in \sqrt{A} \cup A'$. The first equality follows from this and the second equality is clear by assumption with $K = \sqrt{A} \cup A'$.

(ii) Take $c > 0$ sufficiently small such that $[0,c] \cap A = \emptyset$. Let A^* be the reflection of A along the real axis and for suitable $c' > 0$ let $\phi\colon \mathbb{H} \setminus (A \cup A^* \cup [c,\infty)) \to \mathbb{H} \setminus [c',\infty)$ be the unique conformal transformation with

$$\phi(\overline{z}) = \overline{\phi(z)} \quad \text{and} \quad \phi(z)/z \to 1 \text{ and } |z| \to \infty \ .$$

Note that $\Phi_A = \phi$ on $\mathbb{H} \setminus A$. Moreover, $\Phi_{\sqrt{A} \cup A'}(z) = \phi_A(z^2)^{1/2}$ (where we take a branch of the square root takes values in $\mathbb{H}$). The first claim is then immediate and the second part follows directly by taking $\epsilon \to 0$.

(iii) Let γ denote the right boundary of $E([0,\infty))$ and write γ' for the right boundary of $E'([0,\infty)) \cup E''([0,\infty))$. Then for any compact chordal A with $A \cap (-\infty, 0] = \emptyset$,

$$\begin{aligned}
\mathbb{P}(\gamma'([0,\infty)) \cap A = \emptyset) &= \mathbb{P}(E'([0,\infty)) \cap A = \emptyset, E''([0,\infty)) \cap A = \emptyset) \\
&= \mathbb{P}(E'([0,\infty)) \cap A = \emptyset)^2 = \Phi_A'(0)^\alpha \\
&= \mathbb{P}(E([0,\infty)) \cap A = \emptyset) = \mathbb{P}(\gamma([0,\infty)) \cap A = \emptyset) \ .
\end{aligned}$$

The claim follows by Dynkin's lemma.